AN INVESTIGATION OF SINGULAR OPTIMAL CONTROL PROBLEMS

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SUMMARY

The Pontryagin Minimum Principle, when applied to bounded control problems that are linear in the control variable (LOP), explicitly defines only one control function as a candidate for optimality -- the bang-bang control. However, the bang-bang control is undefined when the switching function is identically zero. This property gives rise to the possibility of other controls existing which satisfy the Minimum Principle. It has been shown that for some LOPs there do exist controls and trajectories along which the bang-bang control is not defined. These LOPs have been termed singular since there exist controls (singular controls) other than the bang-bang control which trivially satisfy the Minimum Principle and therefore are candidates for optimality. Consequently, before the LOP can be solved, it must be shown to be non-singular or all the singular controls must be found.

The purpose of this dissertation is to conduct a general investigation of the singular problem in order to develop insight into the nature of singularity and to develop techniques for defining all the singular controls and trajectories.

Several general classes of LOPs are considered which consist of nth order linear, nonlinear, or time-varying systems which are expressible in phase variable form, and a cost functional which is to be minimized along the optimal solution trajectory.

The above LOP is investigated to determine the mechanism producing the singular condition. By considering the phase variable system with

several different cost functionals, singularity is shown to be completely independent of the system dynamics and solely dependent upon the cost functional. Normality is established for several cost functionals while the quadratic cost is shown to cause singularity.

The main body of the thesis is devoted to a detailed analysis of the general LOP with the quadratic cost functional which is shown to be singular by defining all the singular control functions. The characteristic differential equation of the singular controls is derived from which the singular controls are defined and shown to possess the property of "feedback cancellation," i.e., the singular controls actually cancel out the original system dynamics -- linear, nonlinear or time-varying -- and substitute a set of linear stationary dynamics, termed the "effective control."

Since the system being controlled becomes linear and stationary under the application of the singular control, Laplace transform techniques are utilized to derive the analytical expressions for the singular trajectories which are shown to be dense in the state space. The fixed- and free-time singular arcs are respectively characterized by the presence and absence of symmetric eigenvalues in the singular solution, and the realization of free-time singular trajectories is shown to require pole-zero cancellation due to the initial state of the singular arc. The concept of pole-zero cancellation is also used to develop an extremely convenient method for defining the analytical expressions of fixed- and free-time singular arcs of all orders.

The role of the singular arcs in the optimal solution trajectory is discussed, and an example of the regulator problem is solved in detail for both the fixed- and free-time cases.

Finally, a technique for normalizing the singular LOP, which leads to a suboptimal solution, is developed. This technique is applied to a second order example, and the resulting suboptimal solution is shown to compare very favorably with the optimal singular solution.

CHAPTER I

INTRODUCTION

It is the purpose of this thesis to conduct a thorough investigation of the singular problem. The singularity of the linear optimization problem (LOP) will be studied as a function of the cost functional for several general classes of systems. The LOP with a quadratic cost functional will be shown to be singular. For this problem singularity will be characterized, the mechanisms associated with singularity will be demonstrated, and an analytical technique for defining all the singular controls and trajectories will be developed. The role of the singular trajectory in the optimal solution will be discussed. Finally, a technique for normalizing the singular problem, which leads to a suboptimal solution, will be developed.

Definition of the Problem

In the theory of optimal control there are several necessary conditions which can be applied to wide classes of control problems to construct the optimal control function, or optimal controller. These necessary conditions, namely the Pontryagin Minimum Principle, Bellman's Dynamic Programming, and theorems from Calculus of Variation, serve as instruments for selecting the control functions which are candidates for optimality (extremal controls) from the class of admissible controls. In those cases where application of the necessary conditions yields <u>all</u> the possible extremal controls, the optimal control can be selected by comparing the system performance to each extremal control.

There is, however, a class of control problems for which the available necessary conditions do not define all the extremal controls, thereby yielding no conclusive information concerning the nature of the optimal control. This class of control problems has been termed the "singular" optimal control problem, or more simply the "singular" problem.

The General Optimal Control Problem

The basic problem of optimal control is that of determining the control function, u, that will transfer a given system from its initial state to its target set in an optimal fashion, i.e., extremize the performance index or cost functional.

The system (state) equations are ordinary or partial differential equation which represent a mathematical model of the physical system (plant) such as a chemical process, electrical network, sounding rocket, or space vehicle. An accurate model of a physical system may also contain additional constraints such as control and/or state variable constraints. The cost functional is a quantitative measure of the system performance, which may contain a measure of time expended, energy expended, cost, accumulated error, et cetera.

The optimal control problem is mathematically formulated in the following manner. Minimize the cost functional $J[u]$ subject to the following constraints:

(1) The system constraint

$$\dot{\underline{X}} = \underline{f}(\underline{X}, \underline{u}, t) \ ,$$

where $\underline{X}$ and $\underline{f}$ are n vectors, $\underline{u}$ is an r vector, and t is the independent variable usually denoting time.

(2) The control constraint

$$u_i \in \Omega_i(\underline{X},t) \ ,$$

where Ω_i represents the admissible set for u_i, which may be a function of both time and the state variables.

(3) The state variable constraints

$$X_i \in \Upsilon_i$$

where Υ_i represents the allowable region for the state X_i.

(4) The initial and final state constraint

$$\underline{X}(t_0) = \underline{X}_0 \text{ (a fixed point in the state space),}$$

and

$$\underline{X}(t_F) \in \mho(t) \ ,$$

where $\mho(t)$ may be a point or a region in the state space. If $\mho = E^n$ (the entire state space), then a terminal constraint does not exist.

The cost functionals most widely used are due to Bolza and Mayer (1).* Bolza considered the following cost functional

$$J_B[\underline{u}] = \Phi_B[\underline{X}(t_F)] + \int_{t_0}^{t_F} f_0(\underline{X},\underline{u},t)dt \ ,$$

where $\Phi_B[\underline{X}(t_F)]$ is some combination of the final states of the system such

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*Numbers in parentheses following a citation refer to items in the bibliography.

as terminal altitude and/or velocity of a sounding rocket. The integral term $\int_{t_0}^{t_F} f_0(\underline{X},\underline{u},t)dt$ may represent time or energy expended, or accumulated error during the time interval $[t_0,t_F]$.

The Mayer formulation is similar to Bolza's except that f_0 is identically equal to zero, namely

$$J_M[\underline{u}] = \Phi_M[\underline{X}(t_F)] \ .$$

Bolza's formulation can be converted to Mayer's by augmenting the system equations in the following manner. Let

$$\dot{X}_0(t) = f_0(\underline{X},\underline{u},t) \ ,$$

then

$$X_0(t_F) = \int_{t_0}^{t_F} f_0(\underline{X}, \underline{u}, t) \ dt \ ;$$

therefore,

$$J_B[\underline{u}] = \Phi_B[\underline{X}(t_F)] + X_0(t_F) \ ,$$

or,

$$J_B[\underline{u}] = \bar{\Phi}_M[\underline{X}(t_F), \ X_0(t_F)] \ .$$

The Linear Optimization Problem

Johnson and Gibson (2) defined the linear optimization problem (LOP) as that class of optimal control problems in which the control function appears only linearly. It is from this class of problems that the singular problem most commonly arises.

In order to realize any concrete results, it becomes necessary to restrict attention to a more practical sub-class of the general problem described in the previous section. In this thesis, primary concern will be devoted to control problems of the following form:

$$\dot{\underline{X}} = \underline{f}(\underline{X},t) + u\underline{g}(\underline{X},t) \qquad (1\text{-}1)$$

where $\underline{X}$ and $\underline{f}$ are continuous n vectors, and u is a piecewise continuous function. The scalar control function, u, will be constrained to belong to the fixed convex set, Ω, where

$$\Omega = \left\{ u \mid A \leq u \leq B \right\} .$$

State variable constraints will not be considered. The initial conditions will be fixed _a priori_, and the terminal conditions will be either fixed or left unspecified. Cost functionals of the Bolza or Mayer type will be considered; however, if the control appears under the integral, it must appear linearly.

Solution of the LOP

There are several equivalent techniques available for solving optimal control problems. The most important techniques are the Pontryagin Minimum Principle, Bellman's Dynamic Programming and Calculus of Variation. Each of these techniques fails to supply sufficient information for the solution of the LOP. In this chapter, the Pontryagin Minimum Principle is discussed in relation to the LOP. A discussion of the other techniques with respect to the singular problem can be found in (3) and (4).

The Pontryagin Minimum Principle is now applied to the following LOP. Determine the control function, $u(t) \varepsilon \Omega$, that will transfer the system described by

$$\dot{X}_i = f_i(\underline{X}, t) + u g_i(\underline{X}, t) \quad (i = 1, \ldots, n) \tag{1-2}$$

from its initial state, $\underline{X}_0$, to its final state $\underline{X}_F$, and that will minimize the cost functional

$$J[u] = \int_{t_0}^{T} [f_0(\underline{X},t) + u g_0(\underline{X},t)] \, dt , \tag{1-3}$$

where

T, the final time, may be free or fixed,

t_0 is the initial time,

$\underline{X}(t_0) = \underline{X}_0$,

$\underline{X}(T) = \underline{X}_F$, and

$\Omega = \left\{ u \mid A \leq u \leq B \right\}$.

The Pontryagin Minimum Principle represents a necessary condition that each control function must satisfy in order to be a candidate for optimality. It states that the optimal control function must minimize the Hamiltonian (an energy function) with respect to the control. Before defining the Hamiltonian, the system equations will be augmented by the cost functional. Let

$$\dot{X}_0 = f_0(\underline{X}, t) + u g_0(\underline{X}, t) ,$$

then

$$J[u] = X_0(T) .$$

The system equations are now of order $n + 1$. The Hamiltonian is defined as follows:

$$H = \sum_{i=0}^{n} \dot{X}_i P_i \ , \tag{1-4}$$

where the P_i are the costate variables, adjoint variables, or lagrange multipliers defined by the relations

$$\dot{P}_i = - \frac{\partial H}{\partial X_i} \quad \text{for} \quad i = 1,\ldots,n \ , \tag{1-5}$$

which are called the adjoint or costate equations. Equations (1-2) and (1-5) form a set of $2(n+1)$ first order ordinary differential equations which are commonly referred to as the canonical equations. Substituting for $\underline{\dot{X}}$ in equation (1-4) yields

$$H = \sum_{i=0}^{n} [f_i(\underline{X}, t) \ P_i(t)] + u(t) \sum_{i=0}^{n} [g_i(\underline{X},t) \ P_i(t)] \ , \tag{1-6}$$

or,

$$H = \sum_{i=0}^{n} f_i(\underline{X}(t), t) \ P_i(t) + u(t) \ \zeta(\underline{X}(t), \underline{P}(t), t) \ ,$$

where

$$\frac{\partial H}{\partial u} = \zeta(\underline{X}(t), P(t), t) = \sum_{i=0}^{n} g_i(\underline{X}(t), t) \ P_i(t)$$

The Pontryagin Minimum Principle is mathematically stated in the following manner. Let $u(t)$ be an admissible control which transfers

$(\underline{X}(t_0), t_0)$ to $(X(T), T)$. Let $\underline{X}^*(t)$ be the trajectory corresponding to $u^*(t)$, originating at $\underline{X}_0$ and reaching $\underline{X}_F$ (for the first time) at time T. In order that $u^*(t)$ be optimal, it is necessary that there exist a function $P^*(t)$ such that:

1. $P^*(t)$ correspond to $u^*(t)$ and $\underline{X}^*(t)$, so that $P^*(t)$ and $\underline{X}^*(t)$ are a solution of the canonical equations satisfying the boundary conditions

$$\underline{X}^*(t_0) = \underline{X}_0 \quad \text{and} \quad \underline{X}^*(T) = \underline{X}_F \ .$$

2. The Hamiltonian, $H[\underline{X}^*(t), \underline{P}^*(t), u^*(t)]$, has an absolute minimum as a function of $u(t)$ over Ω at $u(t) = u^*(t)$ for t in $[t_0, T]$; that is,

$$\min_{u \varepsilon \Omega} H[\underline{X}^*(t), \underline{P}^*(t), u(t)] = H[\underline{X}^*(t), \underline{P}^*(t), u^*(t)]$$

or, equivalently,

$$H[\underline{X}^*(t), P^*(t), u^*(t)] \leq H[\underline{X}^*(t), \underline{P}^*(t), u(t)]$$

for all $u(t)$ in Ω.

3. The function $H[\underline{X}^*(t), \underline{P}^*(t), u^*(t)]$ is constant for t in $[t_0, T]$; that is,

$$H[\underline{X}^*(t), \underline{P}^*(t), u^*(t)] = H_0 \ ,$$

where $H_0 = 0$ if the final time, T, is free.

Applying condition two of the Minimum Principle to the Hamiltonian normally yields the form of the optimum controller as a function of $\underline{X}(t)$ and $\underline{P}(t)$; however, this is not necessarily the case for the LOP. Consider the Hamiltonian for the LOP:

$$H = \sum_{i=0}^{n} [f_i(\underline{X}(t), t)\ P_i(t)] + u(t)\ \zeta(\underline{X}(t), \underline{P}(t),t) .$$

When the Hamiltonian is explicitly a function of u (that is, $\zeta \not\equiv 0$), the only control function that will minimize H is given by

$$u_B = \begin{cases} A & \text{when } \zeta(\underline{X}(t), \underline{P}(t),t) < 0 \\ B & \text{when } \zeta(\underline{X}(t), \underline{P}(t),t) > 0 \end{cases}$$

This control function must be optimal (if an optimal exists) if there does not exist a $\underline{P}(t)$ and u(t) such that the Hamiltonian is independent of the control, that is $\zeta(\underline{X}(t), \underline{P}(t),t) \equiv 0$ on some measurable time interval. When this is the case, the LOP is termed normal, and the solution is referred to as the bang-bang solution.

The LOP is termed singular if there exist an admissible system trajectory ($\underline{X}_s(t)$) with its corresponding control function ($u_s(t)$) and adjoint vector ($\underline{P}_s(t)$) along which H is explicitly independent of u. Since condition two of the Minimum Principle is trivially satisfied in the singular case, both the singular control (u_s) and the band-bang control (u_B above) must be considered as candidates for optimality. When the LOP is singular, all the singular controls and trajectories must be found before the optimal solution can be determined.

The procedure for solving the LOP via the Minimum Principle can be stated as follows:

First, the Hamiltonian is constructed from which the switching function and the bang-bang control can be defined.

Next, either the LOP must be shown to be normal, or all the admissible singular trajectories with their corresponding controls and adjoint vectors must be found.

Steps one and two correspond to determining all the controls and system subarcs that are candidates for optimality.

Third, from the set of admissible subarcs, all the system trajectories connecting the initial and final states of the system for which there exists an adjoint vector satisfying the Minimum Principle are constructed. These trajectories may be composed of strictly bang-bang, strictly singular, or a combination of bang-bang and singular subarcs.

Finally, the admissible solutions are evaluated on the basis of the cost functional. The solution (or solutions) minimizing the cost functional is selected as the optimal which in turn defines the optimal control function.

History of the Singular Problem

Johnson and Gibson (2) were the first researchers to show that for certain LOPs there do exist controls other than the bang-bang control for which the Minimum Principle can be satisfied. They termed these controls the singular controls and the corresponding LOP the singular problem. They developed a procedure for determining the singular controls and trajectories (if they exist) which is applicable to most 2nd or 3rd-order, linear or nonlinear, free-time LOPs. Through several simple examples, they demonstrated that for certain initial conditions the singular control did, in fact, compose part or all of the optimal control program.

Johnson and Wonham (5) considered a linear, nth-order, phase variable system with a quadratic cost function. For the free-time regulator problem, they developed a somewhat complicated procedure for determining the free-time singular arcs. Also, Wonham proved that the singular arc was the optimal trajectory connecting any two points on the singular arc.

Kopp and Moyer (6), Tait (4), and Kelley (7) all developed an equivalent necessary condition for singular extremals. Using a Green's Theorem approach Tait was able to demonstrate the existence and optimality of fixed-time singular arcs for linear 2nd-order phase variable systems.

The literature contains very little work relating to the actual mechanisms associated with singularity. Also, there does not exist a systematic procedure for defining all the fixed- and free-time singular trajectories in the nth order LOP.

In this research, the singularity of the linear, nonlinear, and time-varying phase variable LOP is shown to be completely independent of the system dynamics and solely a function of the cost functional. The quadratic cost functional is shown to produce singularity in the phase variable LOP. For this problem, singularity is characterized, the mechanisms associated with singularity are demonstrated, and an analytical technique for defining all the singular controls and trajectories is developed. Necessary and sufficient conditions for the existence of both fixed- and free-time singular arcs are derived, and the free-time singular arcs are shown to require pole-zero cancellation due to the initial state of the singular trajectory.

Organizational Outline

In Chapter II, singularity of the completely controllable nth order linear stationary LOP is investigated as a function of the cost functional. This investigation shows that several cost functionals result in normal (non-singular) problems while the LOP with the quadratic cost functional may be singular.

In Chapter III, the singularity of the linear LOP with the quadratic cost functional is established by deriving all the singular control and trajectories that satisfy the Minimum Principle. In addition, singularity is characterized, and the mechanisms associated with the singular trajectories are demonstrated.

The results established in Chapters II and III for the linear system are extended to several classes of nonlinear and time varying systems in Chapter IV. Singularity of the uncontrollable system is also discussed in Chapter IV.

In the preceding chapters, all the control functions that are candidates for optimality, namely the bang-bang control and the singular controls, have been determined for the phase variable LOP with a quadratic cost functional. Using these results, the optimal solution can be found. In Chapter V, the role of the singular arcs in the optimal solution trajectory is discussed. The knowledge of the analytical expressions for the singular arcs is shown to be a useful aid in determining the optimal solution. An example is given.

A technique for normalizing the singular LOP is developed in Chapter VI. Solution of the normalized problem represents a suboptimal solution of the original singular problem which is shown to compare very favorably with the optimal solution.

CHAPTER II

THE LINEAR LOP

In this chapter, the completely controllable linear stationary nth order LOP will be considered. Singularity of this system will be studied as a function of the cost functional. The quadratic cost functional will be shown to produce singularity.

The Linear System

Consider the following linear system

$$\dot{\underline{Z}} = F\,\underline{Z} + \underline{G}\,u \tag{2-1}$$

where

$\underline{Z}$ and $\dot{\underline{Z}}$ are n vectors,

$\underline{G}$ is a constant n vector,

F is a constant n x n matrix, and

u is the scalar control function.

This system is assumed to be completely controllable. Kalman (8) has shown that the above system is completely controllable if and only if the matrix

$$H = [\,\underline{G} \,\vdots\, F\,\underline{G} \,\vdots\, \cdots \,\vdots\, F^{n-1}\underline{G}\,]$$

is non-singular, i.e., the vectors $\underline{G}$, $F\underline{G}$, ..., $F^{n-1}\underline{G}$ are linearly independent and span the n-dimensional vector space.

This system is now transformed into phase variable form following the technique developed by Johnson and Wonham (5). Let

$$\underline{Z} = H\,\underline{Y}\,, \tag{2-2}$$

where H is a non-singular $(n \times n)$ matrix. Substituting $\underline{Z}$ and $\dot{\underline{Z}}$ in (2-1) yields

$$H\,\dot{\underline{Y}} = F\,H\,\underline{Y} + \underline{G}\,u\,. \tag{2-3}$$

Since H is non-singular, (2-3) can be written as

$$\dot{\underline{Y}} = (H^{-1}F\,H)\,\underline{Y} + (H^{-1}\underline{G})u = \overline{F}\,\underline{Y} + \overline{G}\,u\,. \tag{2-4}$$

Straightforward calculations show that the matrix $\overline{F}$ and the vector $\underline{\overline{G}}$ have the following form:

$$\overline{F} = \begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & \overline{f}_1 \\ 1 & 0 & \cdot & \cdot & \cdot & \overline{f}_2 \\ 0 & 1 & \cdot & \cdot & \cdot & \overline{f}_3 \\ \cdot & & \ddots & & & \cdot \\ \cdot & & & \ddots & & \cdot \\ \cdot & & & & \ddots & \cdot \\ 0 & 0 & \cdot & \cdot & 1 & \overline{f}_n \end{bmatrix}, \quad \underline{\overline{G}} = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

where the elements $\overline{f}_i$ in $\overline{F}$ are defined by the relation

$$\overline{f}_i = \underline{h}_i \cdot F\underline{G}$$

for $i = 1,\ldots,n$, where $\underline{h}_i$ is the ith row of H^{-1}. Now, let

$$\underline{X} = \begin{bmatrix} Y_n \\ \dot{Y}_n \\ \cdot \\ \cdot \\ \cdot \\ \overset{n-1}{Y_n} \end{bmatrix} ,$$

then

$$\dot{\underline{X}} = A\,\underline{X} + \underline{b}\,u \ , \tag{2-5}$$

where A and $\underline{b}$ have the form

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 1 & & & & 0 \\ \cdot & \cdot & & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot & & \cdot \\ \cdot & \cdot & & & & \cdot & \cdot \\ 0 & & & & & & 1 \\ \bar{f}_1 & \bar{f}_2 & \cdot & & \cdot & \cdot & \bar{f}_n \end{bmatrix} , \quad \underline{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} .$$

The relation between $\underline{Y}$ and $\underline{X}$ is

$$\underline{Y} = B\,\underline{X} \tag{2-6}$$

where

$$B = \begin{bmatrix} -\bar{f}_2 & -\bar{f}_3 & \cdot & \cdot & \cdot & -\bar{f}_n & 1 \\ -\bar{f}_3 & -\bar{f}_4 & \cdot & \cdot & -\bar{f}_n & 1 & 0 \\ \cdot & & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot \\ -\bar{f}_n & 1 & & & & & \cdot \\ 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \end{bmatrix} .$$

Since B is a triangular matrix with ones on the diagonal,

$$\det(B) = 1 \ ;$$

hence B is non-singular. From (2-2) and (2-6), the relation between $\underline{X}$ and $\underline{Z}$ is determined

$$\underline{Z} = H\ \underline{Y} = H(B\ \underline{X}) = (H\ B)\underline{X} = C\ \underline{X} \ ;$$

therefore,

$$\underline{X}(t_0) = C^{-1}\underline{Z}(t_0) \ .$$

Form (2-5) is the phase variable form for system (2-1), and will be considered the normal form.

Singularity as a Function of the Cost Functional

The LOP for the linear system (2-5) is formulated in the following manner. Find the control function u, where $|u| \leq 1$, that will transfer system (2-5) from $\underline{X}_0$ to $\underline{X}_F$ in an optimum fashion, i.e., minimize the cost functional J[u].

Several different cost functionals will be considered to show that singularity is strictly a function of the cost functional for the completely controllable phase variable linear system. This is not necessarily the case for the uncontrollable system which will be discussed in a later chapter.

The cost functionals considered will be of the Bolza type, i.e.,

$$J[u] = \Phi[\underline{X}(T)] + \int_{t_0}^{T} f_0(\underline{X},t)\ dt \ ,$$

where the final time may be fixed or free. The Hamiltonian is given by

$$H = P_0 f_0 + \sum_{i=1}^{n} P_i \dot{X}_i \tag{2-7}$$

Substituting for $\dot{\underline{X}}$ from (2-5) yields

$$H = P_0 f_0 + \sum_{i=1}^{n-1} P_i X_{i+1} + P_n \sum_{i=1}^{n} a_i X_i + P_n u \; . \tag{2-8}$$

where $a_i = \overline{f}_i$ for $i = 1, \ldots, n$. The adjoint equations are

$$\begin{aligned}
\dot{P}_1 &= -P_0 \frac{\partial f_0}{\partial X_1} - a_1 P_n \\
\dot{P}_2 &= -P_0 \frac{\partial f_0}{\partial X_2} - a_2 P_n - P_1 \\
&\;\;\vdots \\
\dot{P}_n &= -P_0 \frac{\partial f_0}{\partial X_n} - a_n P_n - P_{n-1} \; .
\end{aligned} \tag{2-9}$$

The switching function, $\zeta(\underline{X}, \underline{P})$, is given by

$$\zeta(\underline{X}, \underline{P}) = \frac{\partial H}{\partial u} = P_n(t) \; .$$

Therefore along any singular arc, $P_n(t)$ must be identically zero to make the Hamiltonian independent of u. For a function to be identically zero on some measurable time interval, it is necessary that the function and all its derivatives be zero in that interval. Therefore the singular adjoint equations are formed by setting P_n and $\dot{P}_n$ equal to zero in (2-9), i.e.,

$$\begin{aligned}
\dot{P}_1 &= -P_0 \frac{\partial f_0}{\partial X_1} \\
\dot{P}_2 &= -P_0 \frac{\partial f_0}{\partial X_2} - P_1 \\
&\vdots \\
\dot{P}_n &= -P_0 \frac{\partial f_0}{\partial X_n} - P_{n-1} = 0 .
\end{aligned} \tag{2-10}$$

Therefore,

$$P_{n-1} = -P_0 \frac{\partial f_0}{\partial X_n} . \tag{2-11}$$

<u>Case one</u>

Let $\Phi[\underline{X}(T)] = 0$ and $f_0(\underline{X},t) = \sum_{i=1}^{n} \alpha_i X_i$. Substituting for f_0 in (2-10) yields

$$\begin{aligned}
\dot{P}_1 &= -P_0\alpha_1 \\
\dot{P}_2 &= -P_0\alpha_2 - P_1 \\
&\vdots \\
\dot{P}_n &= -P_0\alpha_n - P_{n-1} = 0 .
\end{aligned}$$

The last equation of (2-12) implies that

$$P_{n-1} = -P_0\alpha_n \quad \text{(a constant)} ,$$

which implies

$$\dot{P}_{n-1} = 0 .$$

Continuing this reasoning yields

$$\dot{P}_{n-1} = 0 \Rightarrow P_{n-2} = -P_0\alpha_{n-1} \Rightarrow \dot{P}_{n-2} = 0 \Rightarrow \ldots \Rightarrow P_1 = -P_0\alpha_2 \Rightarrow$$
$$\dot{P}_1 = 0 \Rightarrow -P_0\alpha_1 = 0 . \qquad (2\text{-}13)$$

Therefore either P_0 or α_1 must be zero. Since $\alpha_1 > 0$, (2-13) implies $P_0 = 0$; but the Minimum Principle requires P_0 to be a positive constant. This contradiction implies singularity cannot exist.

<u>Case two</u>

Let $\Phi[\underline{X}(T)] = 0$ and $f_0(\underline{X},t) = 1$ (the time optimal case). T is free. Substituting for f_0 in (2-10) yields

$$\begin{aligned} \dot{P}_1 &= 0 \\ \dot{P}_2 &= -P_1 \\ &\vdots \\ \dot{P}_n &= -P_{n-1} = 0 . \end{aligned} \qquad (2\text{-}12)$$

This implies

$$P_n = P_{n-1} = \ldots = P_1 = 0 .$$

Since $P_n \equiv 0 \Rightarrow \underline{P} = 0$, the Minimum Principle is violated; therefore, the problem is non-singular.

<u>Case three</u>

Let $f_0(\underline{X},t) = 0$ and remove the restriction on the final state of the system. The final time, T, is fixed. Substituting for f_0 in (2-10) yields

$$\begin{aligned} \dot{P}_1 &= 0 \\ \dot{P}_2 &= -P_1 \\ &\cdot \\ &\cdot \\ &\cdot \\ \dot{P}_n &= -P_{n-1} = 0 \;. \end{aligned} \tag{2-15}$$

As in Case two, (2-15) violates the Minimum Principle; therefore the problem is non-singular.

Case four

Let $\Phi[\underline{X}(T)] = 0$ and $f_0(\underline{X},t) = \sum_{i=1}^{n} Q_i X_i^2$ (the quadratic cost functional). T may be fixed or free. The adjoint equations become

$$\begin{aligned} \dot{P}_1 &= -P_0 Q_1 X_1 \\ \dot{P}_2 &= -P_0 Q_2 X_2 - P_1 \\ &\cdot \\ &\cdot \\ &\cdot \\ \dot{P}_n &= -P_0 Q_n X_n - P_{n-1} = 0 \end{aligned}$$

Upon setting $P_n \equiv 0$ in (2-16), no violation of the Minimum Principle is apparent; therefore this problem may be singular.

Careful examination of the singular adjoint equations in Cases one through four indicates that singularity is not possible unless the canonical equations are coupled. The singular canonical equations will be coupled for system (2-5) only if the cost functional contains nonlinear elements.

The preceding presentation shows that the system parameters (a_i)

do not enter into the singular adjoint equations; therefore, singularity of the linear LOP is not a function of the system dynamics, but is solely dependent upon the cost functional.

CHAPTER III

SINGULARITY OF THE LINEAR LOP WITH THE QUADRATIC COST FUNCTIONAL

This chapter is devoted to the analysis of the linear LOP with the quadratic cost functional. The singularity of this problem is established by defining all the singular arcs and their corresponding singular controls.

The Singular Control Function

The characteristic differential for the singular control function is derived in the following manner for the nth order linear stationary system (2-5) with the quadratic cost functional. The singular state and adjoint equations are

$$\begin{aligned}
\dot{X}_0 &= \frac{1}{2} \sum_{i=1}^{n} Q_i X_i \\
\dot{X}_1 &= X_2 \\
&\ \ \vdots \\
\dot{X}_{n-1} &= X_n \\
\dot{X}_n &= \sum_{i=1}^{n} a_i X_i + u_s \quad ,
\end{aligned} \tag{3-1}$$

and

$$
\begin{aligned}
\dot{P}_0 &= 0 \Rightarrow P_0 = 1 \\
\dot{P}_1 &= -Q_1 X_1 \\
\dot{P}_2 &= -Q_2 X_2 - P_1 \\
&\;\;\vdots \\
\dot{P}_n &= -Q_n X_n - P_{n-1} = 0 \, .
\end{aligned}
\tag{3-2}
$$

where $Q_i \geq 0$ for $i=1,\ldots,n$. Differentiating $\dot{P}_n$ and solving for u_s yields

$$
Q_n u_s = -Q_n \sum_{i=1}^{n} a_i X_i + P_{n-2} + Q_{n-1} X_{n-1} \, . \tag{3-3}
$$

Differentiating u_s $2(n-1)$ times yields

$$
Q_n u_s^{2(n-1)} = -Q_n \sum_{i=1}^{n} a_i X_i^{2(n-1)} + P_{n-2}^{2(n-1)} + Q_{n-1} X_{n-1}^{2(n-1)} \; ; \tag{3-4}
$$

but,

$$
X_{n-1}^{2(n-1)} = X_n^{2n-3} = u_s^{2n-4} + \sum_{i=1}^{n} a_i X_i^{2n-4} = u_s^{2(n-2)} + \sum_{i=1}^{n} a_i X_{n-1}^{n-3+i} \, , \tag{3-5}
$$

and

$$
Q_n \sum_{i=1}^{n} a_i X_i^{2(n-1)} = Q_n \sum_{i=1}^{n} a_i X_n^{n-2+i} = \sum_{i=1}^{n} - a_i P_{n-1}^{n-2+i} = \sum_{i=1}^{n} [a_i Q_{n-1} X_{n-1}^{n-3+i} + a_i P_{n-2}^{n-3+i}] \, . \tag{3-6}
$$

Substituting (3-5) and (3-6) into (3-4) yields

$$Q_n u_s^{2(n-1)} = [-\sum_{i=1}^{n} (a_i Q_{n-1} X_{n-1}^{n-3+i} - a_i P_{n-2}^{n-3+i}] + P_{n-2}^{2(n-1)} + Q_{n-1}[u_s^{2(n-2)} + \sum_{i=1}^{n} a_i X_{n-1}^{n-3+i}] ,$$

which simplifies to

$$Q_n u_s^{2(n-1)} - Q_{n-1} u_s^{2(n-2)} = P_{n-2}^{2n-2} - \sum_{i=1}^{n} a_i P_{n-2}^{n-3+i} ; \qquad (3\text{-}7)$$

but,

$$P_{n-2}^{2n-2} = -Q_{n-2} X_{n-2}^{2n-3} - P_{n-3}^{2n-3} = -Q_{n-2} X_n^{2n-5} - P_{n-3}^{2n-3} = -Q_{n-2} u_s^{2n-6} - \sum_{i=1}^{n} a_i Q_{n-2} X_{n-2}^{n-4+i} - P_{n-3}^{2n-3} , \qquad (3\text{-}8)$$

and

$$\sum_{i=1}^{n} a_i P_{n-2}^{n-3+i} = -\sum_{i=1}^{n} [a_i Q_{n-2} X_{n-2}^{n-4+i} + a_i P_{n-3}^{n-4+i}] \qquad (3\text{-}9)$$

Substituting (3-8) and (3-9) into (3-7) yields

$$Q_n u_s^{2(n-1)} - Q_{n-1} u_s^{2(n-2)} + Q_{n-2} u_s^{2(n-3)} = -P_{n-3}^{2n-3} + \sum_{i=1}^{n} a_i P_{n-3}^{n-4+i} .$$

Repeating these substitutions (n-4) times yields the desired characteristic differential equation for the singular control, namely

$$Q_n u_s^{2(n-1)} - Q_{n-1} u_s^{2(n-2)} + \ldots + (-1)^{n-2} Q_2 \ddot{u}_s + (-1)^{n-1} Q_1 u_s = 0 , \qquad (3\text{-}10)$$

which is a linear ordinary differential equation with constant coefficients.

Now define an "effective control," $\bar{u}$, as

$$\bar{u} = \sum_{i=1}^{n} a_i X_i + u_s \,, \tag{3-11}$$

and derive its characteristic differential equation using (3-1) and (3-2). Differentiating $\dot{P}_n$ and solving for $\bar{u}$ yields

$$Q_n \bar{u} = Q_{n-1} X_{n-1} + P_{n-2} \,. \tag{3-12}$$

Differentiating $\bar{u}$ 2(n-1) times yields

$$Q_n \bar{u}^{2(n-1)} = Q_{n-1} X_{n-1}^{2(n-1)} + P_{n-2}^{2(n-1)} \,,$$

or,

$$Q_n \bar{u}^{2(n-1)} = Q_{n-1} \bar{u}^{2n-4} - Q_{n-2} X_{n-1}^{2n-4} - P_{n-3}^{2n-3} \,,$$

or,

$$Q_n \bar{u}^{2(n-1)} - Q_{n-1} \bar{u}^{2(n-2)} = -Q_{n-2} \bar{u}^{2(n-3)} + Q_{n-3} X_{n-3}^{2n-4} + P_{n-4}^{2n-4} \,,$$

$$\vdots$$

or,

$$Q_n \bar{u}^{2(n-1)} - Q_{n-1} \bar{u}^{2(n-2)} + \ldots + (-1)^{n-3} Q_3 \bar{u}^{4} = (-1)^{n-3} Q_2 \ddot{\bar{u}} + (-1)^{n-3} P_1^{n+1} \,,$$

or,

$$Q_n\bar{u}^{2(n-1)} - Q_{n-1}\bar{u}^{2(n-2)} + \ldots + (-1)^{n-3} Q_3\bar{u}^{4} + (-1)^{n-2} Q_2\ddot{\bar{u}} = (-1)^{n-2} X_1^{n} ;$$

but,

$$X_1^{n} = \dot{X}_n = \bar{u} ;$$

therefore,

$$Q_n\bar{u}^{2(n-1)} - Q_{n-1}\bar{u}^{2(n-2)} + \ldots + (-1)^{n-2} Q_2\ddot{\bar{u}} + (-1)^{n-1} Q_1\bar{u} = 0 \qquad (3\text{-}13)$$

Thus it has been shown that $\bar{u}$ and u_s satisfy the same differential equation. Since the presence of $\underline{a} \cdot \underline{X}$ in $\bar{u}$ did not introduce any additional eigenvalues in $\bar{u}$ that were not already present in u_s, it can be concluded that $\underline{X}_s$ satisfies the same homogeneous differential equation as $\bar{u}$ and u_s, i.e.,

$$Q_n(X_i)_s^{2(n-1)} - Q_{n-1}(X_i)_s^{2(n-2)} + \ldots + (-1)^{n-2} Q_2(\ddot{X}_i)_s + (-1)^{n-1} Q_1(X_i)_s = 0 \qquad (3\text{-}14)$$

for $i=1,\ldots,n$. But the $(X_i)_s$, for $i = 2,\ldots,n$, are formed by differentiating $(X_1)_s$, i.e.,

$$(X_i)_s = (X_1)_s^{i-1} \qquad (3\text{-}15)$$

for $i = 2,\ldots,n$, which implies that $(X_1)_s$ contains all the eigenvalues of the singular system. The singular state equations can therefore be written as

$$X_1^{n} = \bar{u} \tag{3-16}$$

where the solution of (3-16) is required to satisfy (3-14).

The characteristic polynomial of (3-14) is given by

$$Q_n\lambda^{2(n-1)} - Q_{n-1}\lambda^{2(n-2)} + \dots + (-1)^{n-2} Q_2\lambda^2 + (-1)^{n-1} Q_1 = 0 . \tag{3-17}$$

Assume $Q_r > 0$ and $Q_{r+1} = \dots = Q_n = 0$ where $r \in \{1,\dots,n\}$, then the solution of (3-17) contains (r-1) pairs of symmetric* eigenvalues, namely

$$\{\pm\lambda_1, \pm\lambda_2, \dots, \pm\lambda_{r-1}\} \tag{3-18}$$

The eigenvalues of X_1 and $\bar{u}$ are restricted to belong to this set, where the λ_i are not restricted to be either real or distinct.

Kopp and Moyer (6), Tait (4), and Kelley (7) all developed an equivalent necessary condition for singular extremals given by

$$V = (-1)^K \frac{\partial}{\partial u}\left\{\frac{d^{2K}}{dt^{2K}}\left(\frac{\partial H}{\partial u}\right)\right\} \geq 0 . \tag{3-19}$$

If $V = 0$ for $K = 1,\dots, m-1$ and $V > 0$ for $K = m$, then the associated singular arc satisfies the necessary condition, and the problem is referred to as an mth-order singular problem. Calculations show that (3-19) is satisfied if $Q_i \geq 0$. If $Q_r > 0$ and $Q_{r+1} = \dots = Q_n = 0$, then the singularity is of order $(n+1-r)$, and (3-18) contains [n-(order of singularity)] pairs of symmetric eigenvalues or $[n - (n+1-r)] = r-1$.

---

*Two eigenvalues are symmetric if they are equal in magnitude and opposite in phase.

Equations (3-10) and (3-13) imply that $\bar{u}$ can be expressed as a linear combination of the state variables, i.e.,

$$\bar{u} = \sum_{i=1}^{n} b_i X_i \ . \tag{3-20}$$

Since the singular state equation is of order n, the b_i are chosen to produce the desired n eigenvalues in the solution of (3-16). There corresponds a unique $\underline{b}$ for each set of n eigenvalues selected from (3-18). Complex eigenvalues must be selected in conjugate pairs so that $\underline{b}$ is a real constant vector.

Assume that n desired eigenvalues are selected and that they form the following characteristic polynomial

$$(S - \lambda_1)(\ldots)(S - \lambda_n) = S^n + c_n S^{n-1} + \ldots + c_2 S + c_1 = 0 \tag{3-21}$$

The characteristic equation for (3-16) is given by

$$S^n - b_n S^{n-1} - \ldots - b_2 S - b_1 = 0 \tag{3-22}$$

Comparing (3-21) and (3-22), one finds

$$b_i = -c_i \ , \tag{3-23}$$

for i = 1,...,n. Since $\underline{b}$ is unique, the singular control is uniquely defined by (3-11), i.e.,

$$u_s = \bar{u} - \underline{a} \cdot \underline{X} \ ,$$

where,

$$\bar{u} = \sum_{i=1}^{n} b_i X_i = - \sum_{i=1}^{n} c_i X_i = -\underline{c} \cdot \underline{X}$$

Thus,

$$u_s = - \underline{a} \cdot \underline{X} - \underline{c} \cdot \underline{X} .$$

This form for the singular control illustrates one of the characteristics of the singular control, namely that of "feedback cancellation." The singular control effectively reshapes the feedback structure of the original system so that the singular system will function in an optimal fashion.

The Singular Adjoint Variables

In order for the singular control, derived in the preceding section, to satisfy the Minimum Principle, there must exist an adjoint vector such that the Hamiltonian is a constant. The singular adjoint variables are derived in the following manner. From (3-2), it is known that

$$\begin{aligned} P_{n-1} &= -Q_n X_n , \\ P_{i-1} &= -\dot{P}_i - Q_i X_i \end{aligned} \qquad (3\text{-}24)$$

for $i = 2,\ldots,n-1$, and

$$\dot{P}_1 = -Q_i X_i .$$

Therefore,

$$P_{n-2} = -\dot{P}_{n-1} - Q_{n-1} X_{n-1} = Q_n \overset{1}{X}_n - Q_{n-1} \overset{-1}{X}_n$$

where $\overset{-1}{X}_n = \int X_n dt = X_{n-1}$ and $\overset{1}{X}_n = \dot{X}_n$. Similarly,

$$P_{n-3} = -\dot{P}_{n-2} - Q_{n-2}X_{n-2} = -Q_n X_n^{2} + Q_{n-1}X_n^{0} - Q_{n-2}X_{n-2}^{-2}$$

$$\vdots$$

$$P_{n-i} = (-1)^i \left\{ Q_n X_n^{i-1} - Q_{n-1}X_n^{i-3} + \ldots + (-1)^{i-2} Q_{n-(i-2)}X_n^{-(i-3)} + (-1)^{i-1} Q_{n-(i-1)}X_n^{-(i-1)} \right\}$$

$$= (-1)^i \sum_{j=1}^{i} (-1)^{j-1} Q_{n-(j-1)}X_n^{i+1-2j} \tag{3-25}$$

$$\vdots$$

$$P_1 = (-1)^{n-1} \sum_{j=1}^{n-1} (-1)^{j-1} Q_{n-(j-1)} X_n^{n-2j} .$$

The Hamiltonian

The Hamiltonian along the singular trajectories is given by

$$H = \frac{1}{2} \sum_{i=1}^{n} Q_i X_i^2 + \sum_{i=1}^{n-1} P_i X_{i+1} . \tag{3-26}$$

The Minimum Principle requires the Hamiltonian to be a constant along an optimal singular arc. This requirement is satisfied if and only if $\frac{dH}{dt} = 0$.

Differentiating (3-26) with respect to t yields

$$\frac{dH}{dt} = \sum_{i=1}^{n} Q_i X_i \dot{X}_i + \sum_{i=1}^{n-1} \dot{P}_i X_{i+1} + \sum_{i=1}^{n-1} P_i \dot{X}_{i+1} . \tag{3-27}$$

But from (3-1) and (3-2), it is known that

$$\left.\begin{aligned} \dot{X}_i &= X_{i+1} \quad (i=1,\ldots,n-1), \\ \dot{P}_1 &= -Q_1X_1 , \\ \dot{P}_i &= -Q_iX_i - P_{i-1} \quad (i=2,\ldots,n-1) . \end{aligned}\right\} \tag{3-28}$$

and

Substituting in (3-27) yields

$$\frac{dH}{dt} = \sum_{i=1}^{n} Q_iX_iX_{i+1} - Q_1X_1X_2 + \sum_{i=2}^{n-1} (-Q_iX_i - P_{i-1})X_{i+1} + \sum_{i=1}^{n-1} P_iX_{i+2} ,$$

which reduces to

$$\frac{dH}{dt} = Q_nX_nX_{n+1} - \sum_{i=2}^{n-1} P_{i-1}\, X_{i+1} + \sum_{i=1}^{n-1} P_i\, X_{i+2} ,$$

or,

$$\frac{dH}{dt} = Q_nX_nX_{n+1} + P_{n-1}X_{n+1} ; \tag{3-29}$$

but,

$$P_{n-1} = -Q_nX_n .$$

Substituting for P_{n-1} in (3-29) yields

$$\frac{dH}{dt} = Q_nX_nX_{n+1} + (-Q_nX_n)X_{n+1} = 0$$

for all time during which the singular condition is maintained. Therefore, all the singular arcs satisfy the Minimum Principle and are candidates as optimal trajectories.

The actual form of the Hamiltonian along the singular arcs is now derived using (3-25). From (3-26),

$$H = \frac{1}{2} \sum_{i=1}^{n} Q_i X_i^2 + \sum_{i=1}^{n-1} P_{n-i} X_{n+1-i} \,. \tag{3-30}$$

Substituting for P_{n-i} from (3-25) yields

$$H = \frac{1}{2} \sum_{i=1}^{n} Q_i X_i^2 + \sum_{i=1}^{n-1} (-1)^i X_{n+1-i} \sum_{j=1}^{i} (-1)^{j+1} Q_{n-(j-1)} X_n^{i+1-2j} , \tag{3-31}$$

or,

$$H = \frac{1}{2} \sum_{i=1}^{n} X_i^2 + \sum_{i=1}^{n-1} (-1)^i \sum_{j=1}^{i-1} (-1)^{j-1} Q_{n-(j-1)} X_n^{i+1-2j} X_n^{-(i-1)} - \sum_{i=1}^{n-1} Q_{n-(i-1)} \left(X_n^{-(i-1)}\right)^2 \tag{3-32}$$

but,

$$\sum_{i=1}^{n-1} Q_{n-(i-1)} \left(X_n^{-(i-1)}\right)^2 = \sum_{i=2}^{n} Q_i X_i^2 ; \tag{3-33}$$

therefore (3-32) reduces to

$$H = \frac{1}{2} Q_1 X_1^2 - \frac{1}{2} \sum_{i=2}^{n} Q_i X_i^2 + \sum_{i=1}^{n-1} (-1)^i X_n^{-(i-1)} \sum_{j=1}^{i-1} (-1)^{j-1} Q_{n-(j-1)} X_n^{i+1-2j} ,$$

or,

$$H = \frac{1}{2} Q_1 X_1^2 - \frac{1}{2} \sum_{i=2}^{n} Q_i X_i^2 + \sum_{i=2}^{n-1} (-1)^{n+1-i} X_i \sum_{j=1}^{n-i} A_j \ , \tag{3-34}$$

where

$$A_j = (-1)^{j-1} Q_{n-(j-1)} X_n^{n+2-i-2j} \ . \tag{3-35}$$

Noting that

$$X_n^{i} = (\dot{X}_n)^{i-1} = \bar{u}^{i-1} \ ,$$

and substituting in (3-35) yields

$$\sum_{j=1}^{n-i} A_j = \sum_{j=1}^{K} (-1)^{j-1} Q_{n-(j-1)} \bar{u}^{n+1-i-2j} + \sum_{j=K+1}^{n-i} (-1)^{j-1} Q_{n-(j-1)} X_{2n+2-i-2j}$$

where $K = \begin{cases} \frac{n-i}{2} & \text{when } (n-i) \text{ is even} \\ \frac{n-i+1}{2} & \text{when } (n-i) \text{ is odd} \end{cases}$.

Since $\bar{u}$ is a linear combination of the states,

$$(-1)^{n+1-i} \sum_{j=1}^{n-i} A_j = \beta_{1i} X_1 + \dots + \beta_{ni} X_n = \beta_i \cdot \underline{X} \ . \tag{3-36}$$

Substituting in (3-34) yields

$$H = \frac{1}{2} Q_1 X_1^2 - \frac{1}{2} Q_n X_n^2 - \frac{1}{2} \sum_{i=2}^{n-1} Q_i X_i^2 + \sum_{i=2}^{n-1} X_i \underline{\beta}_i \cdot \underline{X} \ , \tag{3-37}$$

or,

$$H = \frac{1}{2} Q_1 X_1^2 - \frac{1}{2} Q_n X_n^2 - \frac{1}{2} \sum_{i=2}^{n-1} (Q_i - \beta_{ii}) X_i^2 + \sum_{i=2}^{n-1} \sum_{\substack{j=1 \\ j \neq i}}^{n} \beta_{ji} X_j X_i ,$$

or,

$$H = \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij} X_i X_j = \underline{X} \cdot B\underline{X} \tag{3-38}$$

Since the Hamiltonian is equal to a constant along the entire singular trajectory, i.e.,

$$H(t) = H_o \quad \text{(a constant)},$$

at time $t = t_s$, where t_s is the time at which the singular control is applied, H must equal H_0.

Let

$$\underline{X}(t_s) = \underline{X}_0 ,$$

then

$$H_0 = \frac{1}{2} Q_1 X_{10}^2 - \frac{1}{2} Q_n X_{n0}^2 - \frac{1}{2} \sum_{i=2}^{n-1} (Q_i - \beta_{ii}) X_{i0}^2 + \sum_{i=2}^{n-1} \sum_{\substack{j=1 \\ j \neq i}}^{n} \beta_{ji} X_{j0} X_{i0} . \tag{3-39}$$

This implies that the value of the Hamiltonian along the singular arc is only a function of the position of the states at the instant the singular control is applied. This fact gives rise to the possibility of both fixed- and free-time singular arcs existing in this singular problem.

Along a fixed-time singular arc H_0 must be a non-zero constant, while $H_0 = 0$ along a free-time singular arc. The equation of the fixed- or free-time singular trajectory is given by (3-38) and (3-39) as

$$\underline{X} \cdot B\underline{X} = H_0 \, . \tag{3-40}$$

A more useful expression is derived in the following section.

The Singular Trajectory

In the preceding sections, the singular trajectory (arc) was shown to be that system trajectory resulting from the application of an admissible singular control. The Hamiltonian along this arc was shown to be a constant, H_0, a function of the initial singular state of the system only. Since the value of H_0 determines whether the singular arc is optimal for a fixed- or free-time problem, H_0 will be related to the parameters of the singular solution.

Consider an nth order singular problem with the following set of 2(n-1) distinct, real eigenvalues.

$$\left\{\pm \lambda_1, \ \pm \lambda_2, \ldots, \ \pm \lambda_{n-1}\right\} . \tag{3-41}$$

Select

$$\left\{+ \lambda_1, \ + \lambda_2, \ldots, \ + \lambda_{n-1}, \ - \lambda_1\right\} \tag{3-42}$$

to be the desired eigenvalues of the singular system. Assume the characteristic polynomial corresponding to (3-42) is

$$S^n + c_n S^{n-1} + c_{n-1} S^{n-2} + \ldots + c_2 S + c_1 = 0 \, .$$

Using Laplace transforms to solve for X_1, yields

$$X_1(S)[S^n + c_n S^{n-1} + \ldots + c_2 S + c_1] = \tag{3-43}$$
$$X_{10}S^{n-1} + S^{n-2}(X_{20} + c_n X_{10}) + \ldots + (X_{n0} + c_n X_{(n-1)0} + \ldots + c_2 X_{10})$$

where,

$$X_{i0} = \overset{i-1}{X_{10}} = \overset{i-1}{X_1}(t_s) .$$

Solving for $X_1(S)$ yields

$$X_1(S) = \frac{X_{10}S^{n-1} + (X_{20} + c_n X_{10})S^{n-2} + \ldots + (X_{n0} + c_n X_{(n-1)0} + \ldots + c_2 X_{10})}{(S-\lambda_1)(S-\lambda_2)(\ldots)(S-\lambda_{n-1})(S-\lambda_n)} = \frac{N(S)}{D(S)} ,$$

where $\lambda_n = -\lambda_1$, or,

$$X_1(S) = \frac{A_1}{S-\lambda_1} + \frac{A_2}{S-\lambda_2} + \ldots + \frac{A_{n-1}}{S-\lambda_{n-1}} + \frac{A_n}{S-\lambda_n} , \tag{3-44}$$

where,

$$A_i = A(\lambda_1) = \frac{N(\lambda_i)}{\left[\frac{D(S)}{(S-\lambda_i)}\right]_{S=\lambda_i}} = \frac{N(\lambda_i)}{F(\lambda_i)} \tag{3-45}$$

for i=1,...,n-1, and

$$A_n = A(\lambda_n) = \frac{N(\lambda_n)}{F(\lambda_n)} = \frac{N(-\lambda_1)}{F(-\lambda_1)} = A(-\lambda_1) = A_{-1} .$$

Therefore, $X_1(t)$ can be written as

$$X_1(t) = \sum_{i=1}^{n} A_i \, e^{\lambda_i t} \; . \qquad (3\text{-}46)$$

The remaining states are obtained by successive differentiation of $X_1(t)$, i.e.,

$$X_i(t) = X_1^{i-1}(t) = \sum_{j=1}^{n} (\lambda_j)^{i-1} A_j \, e^{\lambda_j t} \qquad (3\text{-}47)$$

for $i = 2,\ldots,n$.

Equation (3-38) shows that the Hamiltonian is equal to some particular combination of the states squared and the product of the states taken two at a time, i.e.,

$$H = \sum_{i=1}^{n} \sum_{j=1}^{n} B_{ij} X_i(t) X_j(t) \;, \qquad (3\text{-}48)$$

where B_{ij} is a function of Q. From (3-46) and (3-47), it is clear that (3-48) may be rewritten as

$$H = \sum_{i=1}^{n} \sum_{j=1}^{n} B_{ij} \left\{ \lambda_1^{i+j-2} A_1^2 \, e^{2\lambda_1 t} + \ldots + \lambda_n^{i+j-2} A_n^2 \, e^{2\lambda_n t} + \sum_{K=1}^{n} \sum_{\substack{L=1 \\ L \neq K}}^{n} \lambda_K^{i-1} \lambda_L^{j-1} A_K A_L e^{(\lambda_K + \lambda_L)t} \right\} . \qquad (3\text{-}49)$$

Since H is a constant, each time dependent term in (3-49) must have a zero coefficient. The only terms in (3-49) independent of time are of the form

$$K\, e^{(\lambda_1+\lambda_n)t}$$

since $\lambda_n = -\lambda_1$. Collecting all the constant terms in H yields

$$H = \sum_{i=1}^{n} \sum_{j=1}^{n} B_{ij}\left\{\lambda_1^{i-1}\lambda_n^{j-1} A_1A_n + \lambda_n^{i-1}\lambda_1^{j-1} A_nA_1\right\}, \tag{3-50}$$

or,

$$H = A_1A_n \sum_{i=1}^{n} \sum_{j=1}^{n} B_{ij}\lambda_1^{i+j-2}\left\{(-1)^{j-1} + (-1)^{i-1}\right\} = K\, A_1A_n ,$$

where,

$$A_1 = \frac{N(\lambda_1)}{F(\lambda_1)} \quad \text{and} \quad A_n = A_{-1} = \frac{N(-\lambda_1)}{F(-\lambda_1)} .$$

Substituting for A_1 and A_n yields

$$H = \frac{K}{F(\lambda_1)F(-\lambda_1)} N(\lambda_1)N(-\lambda_1) ,$$

where,

$$N(\lambda) = X_{10}\lambda^{n-1} + (X_{20} + c_nX_{10})\lambda^{n-2} + \ldots + (X_{no} + c_nX_{(n-1)0} + \ldots + c_2X_{10}),$$

which may be rewritten as

$$\begin{aligned} N(\lambda) = X_{n0} &+ X_{(n-1)0}(c_n+\lambda) + X_{(n-2)0}(c_{n-1}+\lambda c_n+\lambda^2) \\ &+ \ldots + X_{10}(c_2+\lambda c_3+\lambda^2c_4 + \ldots + \lambda^{n-2}c_n + \lambda^{n-1}) . \end{aligned} \tag{3-51}$$

The terms $N(\lambda_1)$ and $N(-\lambda_1)$ are both constants which are functions of the initial state of the singular arc, $\underline{X}_0$. Therefore the constant value of the Hamiltonian, H_0, is given by

$$H_0 = \frac{K}{F(\lambda_1)F(-\lambda_1)} N(\lambda_1)N(-\lambda_1) \ . \tag{3-52}$$

The Hamiltonian as a function of $\underline{X}$ is formed by replacing $\underline{X}_0$ by $\underline{X}$ in (3-52), i.e.,

$$H = \frac{K}{F(\lambda_1)F(-\lambda_1)} \bar{N}(\lambda_1)\bar{N}(-\lambda_1) = H_0 \tag{3-53}$$

where

$$\bar{N}(\lambda) = N(\lambda)\Big|_{\underline{X}_0 \leftarrow \underline{X}} \ .$$

Comparing the coefficients of X_{no}^2 in (3-39) and (3-53), yields

$$\frac{K}{F(\lambda_1)F(-\lambda_1)} = -\frac{1}{2} Q_n \ . \tag{3-54}$$

From (3-53) and (3-54), the equations of the singular trajectories is found to be

$$Q_n\bar{N}(\lambda_1)\bar{N}(-\lambda_1) = -2H_o \ ,$$

or,

$$Q_n[X_n + X_{n-1}(c_n + \lambda_1) + \dots + X_1(c_2 + \lambda_1 c_3 + \dots + \lambda_1^{n-2}c_n + \lambda_1^{n-1}] \tag{3-55}$$

$$[X_n + X_{n-1}(c_n - \lambda_1) + \dots + X_1(c_2 - \lambda_1 c_3 + \dots + (-\lambda_1)^{n-2}c_n + 1 - \lambda_1)^{n-1}] = -2H_0 \ .$$

This is the equation of the nth order singular trajectories corresponding to the selected set of n eigenvalues which contained only one pair of symmetric eigenvalues.

Now consider a set of n eigenvalues which contains r pairs of symmetric eigenvalues, namely

$$\left\{+\lambda_1, +\lambda_2, \ldots, +\lambda_{n-r}, -\lambda_1, \ldots, -\lambda_r\right\} \tag{3-56}$$

where $(n-r) \geq r > 1$. (This must be the case when the order of singularity equals r, since every set of n eigenvalues selected from a set containing (n-r) pairs of symmetric eigenvalues must contain at least r pairs of symmetric eigenvalues.) Let A_i (for $i = 1,\ldots,n-r$) be the residue of the pole $(S - \lambda_i)$ in $X_1(S)$, and let A_{-i} (for $i = 1,\ldots,r$) be the residue of $(S + \lambda_i)$, where

$$A_i = \frac{N(\lambda_i)}{F(\lambda_i)} \quad \text{and} \quad A_{-i} = \frac{N(-\lambda_i)}{F(-\lambda_i)} \,.$$

Then, in a similar manner, H can be written as

$$H = \sum_{i=1}^{r} \frac{K_i}{F(\lambda_i)F(-\lambda_i)} \bar{N}(\lambda_i)\bar{N}(-\lambda_i) = H_0 = \sum_{i=1}^{r} \frac{K_i}{F(\lambda_i)F(-\lambda_i)} N(\lambda_i)N(-\lambda_i). \tag{3-57}$$

In the preceding development, the admissible set of eigenvalues was assumed to contain 2(n-1) real, distinct eigenvalues. This restriction was imposed for the sole purpose of simplifying the presentation. Similar results are obtained when this restriction is removed.

The primary result of this section may be summarized by the following theorem.

Theorem I. Consider system (2-5) with the quadratic cost functional. Assume (3-18) contains at least n eigenvalues. If the selected set of n eigenvalues contains r pairs of symmetric eigenvalues, then the singular trajectories are given by

$$\bar{K}_1\bar{N}(\lambda_1)\bar{N}(-\lambda_1) + \ldots + \bar{K}_r\bar{N}(\lambda_r)\bar{N}(-\lambda_r) = H_0 \qquad (3\text{-}58)$$

where

$$\bar{K}_i = \frac{K_i}{F(\lambda_i)F(-\lambda_i)} \neq 0 ,$$

and

$$H_0 = \sum_{i=1}^{r} \bar{K}_i N(\lambda_i)N(-\lambda_i) \qquad (3\text{-}59)$$

which is a function of the initial state of the singular arc.

The singular trajectories described by (3-58) are fixed-time arcs if $H_0 \neq 0$, and free-time singular arcs if $H_0 = 0$. These two cases are now characterized in more detail, and the concept of pole-zero cancellation is used to define reduced order singular arcs.

Free-time Singular Arcs

Optimal control problems not containing a final time constraint are referred to as free-time problems. For these problems, the Minimum Principle requires the Hamiltonian to be identically zero along an optimal system trajectory. Therefore, singular arcs for which $H_0 = 0$ are called free-time singular arcs.

In the preceding section, the constant value of the Hamiltonian,

H_0, was shown to consist of up to n/2 terms, each term being a function of the initial state of the singular arc and resulting from the presence of a pair of symmetric eigenvalues in the singular solution. The nature of H_0 is now examined in detail to determine the conditions required for the existence of free-time singular arcs.

Consider a singular problem whose admissible set of eigenvalues contains (n-1) pairs of symmetric eigenvalues -- the first order singular problem. Every set of n eigenvalues selected contains at least one pair of symmetric eigenvalues. Assume the selected set contains only one pair of symmetric eigenvalues. Then, from Theorem I, H_0 is given by

$$H_0 = \bar{K}_1 N(\lambda_1)\ N(-\lambda_1)\ , \qquad (3\text{-}60)$$

where $\bar{K}_1$ is a non-zero constant dependent solely upon the selected set, and the $N(\lambda)$ are functions of $\underline{X}_0$, the initial state of the singular arc. Clearly, $H_0 = 0$ if and only if

$$N(\lambda_1) = 0 \quad \text{and/or} \quad N(-\lambda_1) = 0\ .$$

Recall that

$$N(\lambda) = A(\lambda)\ F(\lambda)\ ,$$

where $F(\lambda)$ is a non-zero constant and $A(\lambda)$ is the residue of the pole $(S-\lambda)$ in the singular solution of $X_1(S)$. Therefore, $N(\lambda) = 0$ if and only if $A(\lambda) = 0$. But, $A(\lambda) = 0$ implies that the eigenvalue λ has been removed from the singular solution of $X_1(t)$. In other words, $N(\lambda) = 0$ if pole-zero cancellation (due to the initial singular conditions) occurs in such a way as to cancel the pole $(S-\lambda)$ in $X_1(S)$. From this it is clear that at least one of the symmetric poles must be cancelled for H_0 to be equal to zero.

When the selected set of n singular eigenvalues contains r pairs of symmetric eigenvalues, H_0 is given by Theorem I as

$$H_0 = \sum_{i=1}^{r} \bar{K}_i \; N(\lambda_i) \; N(-\lambda_i) \; .$$

In this case, $H_0 = 0$ if and only if

$$N(\lambda_i) = 0 \quad \text{and/or} \quad N(-\lambda_i) = 0$$

for $i = 1,\ldots,r$; i.e., at least one of each pair of symmetric poles must be cancelled due to the initial state of the singular arc. Note that when the singular problem is of order r, there exist only (n-r) pairs of admissible eigenvalues, and every set of n eigenvalues selected must contain at least r pairs of symmetric eigenvalues.

The free-time singular arcs are characterized by the following theorem.

Theorem II. In order that the singular trajectories given by Theorem I satisfy the Minimum Principle in the free-time case it is necessary and sufficient that

(1) $X_1(t)$ contain only eigenvalues belonging to the admissible set, and

(2) $X_1(t)$ does not contain any pairs of symmetric eigenvalues.

Note: In order for conditions (1) and (2) to be satisfied simultaneously, it is necessary for pole-zero cancellation (due to the initial state of the singular arc) to occur in such a manner as to cancel at least one of each pair of symmetric poles in $X_1(S)$.

<u>Definition</u>. A singular trajectory is defined to be an mth order singular arc if $X_1(t)$ contains m eigenvalues.

The equation of the nth order singular arcs were given in Theorem I. The free-time singular arcs were shown to require pole-zero cancellation to insure that $X_1(S)$ does not contain any symmetric poles. Clearly, the order of the singular arc is reduced by one for every pole that is cancelled. Therefore, the mth order singular problem admits free-time singular arcs of order (n-m) and less. The equation of the free-time singular arcs is now shown to have a particularly simple form. Consider a free-time singular arc of order (n-1) satisfying Theorem II. Assume the pole $(S - \lambda_n)$ was cancelled which implies $A(\lambda_n) = H_0 = 0$. The singular trajectory is given by (3-58) as

$$\bar{N}(\lambda_n)\ \bar{N}(-\lambda_n) = N(\lambda_n)\ N(-\lambda_n)\ ; \tag{3-61}$$

but, $N(\lambda_n) = 0$. Therefore, (3-61) becomes

$$\bar{N}(\lambda_n)\ \bar{N}(-\lambda_n) = 0 \tag{3-62}$$

for all t during the application of the singular control. Since $\bar{N}(\lambda_n)\Big|_{t=t_s} = N(\lambda_n) = 0$, (3-62) implies that the (n-1)st order free-time singular trajectory is simply given by

$$\bar{N}(\lambda_n) = 0\ , \tag{3-63}$$

an (n-1)st order hyperplane in the state space which is called the singular surface.

Similarly, if a singular arc resulted from the cancellation of r poles, say $(S - \lambda_1),\ldots,(S-\lambda_r)$, then the equation of the (n-r)th order singular arc is given by the simultaneous solution of $N(\lambda_1) = 0$, $N(\lambda_2) = 0,\ldots,$ and $N(\lambda_r) = 0$. In this case, the singular arcs lie on an (n-r)th order hyperplane. The motion of the singular states on the singular surface is a function of the remaining eigenvalues in the singular solution and the point of entry onto the singular surface.

Theorem II shows that pole-zero cancellation is required for the existence of free-time singular arcs. It is now shown that a cancelled pole in no way affects the singular trajectory or the value of the singular control function on the singular surface. Consider a first order singular problem whose (n-1)st order free-time singular trajectory is a function of the following (n-1) eigenvalues selected from the admissible set.

$$\left\{\lambda_1, \lambda_2, \ldots, \lambda_{n-1}\right\} \tag{3-64}$$

where (3-64) does not contain a symmetric pair. Since the singular state equation is nth order, it will have n eigenvalues. Assume the nth eigenvalue is arbitrarily selected to be λ_n. Then the characteristic polynomial for the singular state equation is given by

$$\begin{aligned}[(S-\lambda_1)(\ldots)(S-\lambda_{n-1})](S-\lambda_n) &= [S^{n-1} + A_{n-1}S^{n-2} + \ldots + A_1](S-\lambda_n) \\ &= S^n + c_n S^{n-1} + \ldots + c_2 S + c_1 \ ,\end{aligned} \tag{3-65}$$

where

$$c_n = A_{n-1} - \lambda_n ,$$

$$c_1 = -\lambda_n A_1 ,$$

and

$$c_i = A_{i-1} - \lambda_n A_i \quad (i=2,\ldots,n-1) ,$$

which implies

$$X_1(S) = \frac{S^{n-1}X_{10} + S^{n-2}(X_{20} + c_n X_{10}) + \ldots + (X_{n0} + c_n X_{(n-1)0} + \ldots + c_2 X_{10})}{(S - \lambda_1)(\ldots)(S - \lambda_{n-1})(S - \lambda_n)} \quad (3\text{-}66)$$

The pole $(S - \lambda_n)$ is cancelled if and only if $N(\lambda_n) = 0$. Calculating $N(\lambda_n)$ from (3-51) yields

$$N(\lambda_n) = X_{n0} + A_{n-1}X_{(n-1)0} + \ldots + A_2X_{20} + A_1X_{10} . \quad (3\text{-}67)$$

When $N(\lambda_n) = 0$, $X_1(S)$ can be written as

$$X_1(S) = \frac{[S^{n-2}X_{10} + S^{n-3}(X_{20} + A_{n-1}X_{10}) + \ldots + (X_{(n-1)0} + A_{n-1}X_{(n-2)0} + \ldots A_2X_{10})]\cancel{(S-\lambda_n)}}{(S-\lambda_1)(\ldots)(S - \lambda_{n-1})\cancel{(S-\lambda_n)}},$$

which shows that the singular solution is independent of the cancelled pole. The singular control u_s corresponding to the n selected eigenvalues is given by (3-11) as

$$u_s = \bar{u} - \underline{a} \cdot \underline{X} = \underline{b} \cdot \underline{X} - \underline{a} \cdot \underline{X} ,$$

where

$$b_n = -c_n = -A_{n-1} + \lambda_n ,$$

$$b_1 = -c_1 = \lambda_n A_1 ,$$

and

$$b_i = -c_i = -A_{i-1} + \lambda_n A_i$$

for i = 2,...,n-1. The region of admissibility of the singular surface ($\bar{N}(\lambda_n) = 0$) is determined solely by the control constraint, $|u_s| \leq 1$. Consider the value of $\bar{u}$ on the singular surface, i.e.,

$$\bar{u} = \sum_{i=1}^{n} b_i X_i = \lambda_n A_1 X_1 + (-A_1 + \lambda_n A_2)X_2 + \ldots + (-A_{n-2} + \lambda_n A_{n-1})X_{n-1} + (-A_{n-1} + \lambda_n)X_n \ ;$$

but, since $\bar{N}(\lambda_n) = 0$,

$$A_1 X_1 = -A_2 X_2 - \ldots - A_{n-1} X_{n-1} - X_n \ ,$$

and $\bar{u}$ becomes

$$\bar{u} = -\lambda_n \sum_{i=2}^{n-1} A_i X_i - \lambda_n X_n - \sum_{i=1}^{n-1} A_i X_{i+1} + \lambda_n \sum_{i=2}^{n-1} A_i X_i + \lambda_n X_n$$

$$= - \sum_{i=1}^{n-1} A_i X_{i+1} \ .$$

Therefore u_s is independent of the cancelled pole $(S - \lambda_n)$ on the singular surface.

Remark. In the preceding development, it was shown that when the product of the desired poles were given by

$$S^{n-1} + A_{n-1}S^{n-2} + \ldots + A_2 S + A_1 \ ,$$

the singular surface resulting from the cancellation of the nth pole was given by

$$X_n + A_{n-1}X_{n-1} + \dots + A_2X_2 + A_1X_1 = 0 .$$

Inspection of these equations indicates an extremely simple method for calculating $\overline{N}(\lambda)$ and consequently the singular surface, $\overline{N}(\lambda) = 0$. Similarly, when the singular surface is a function of (n-r) poles, say

$$(S - \lambda_1)(\dots)(S - \lambda_{n-r}) = S^{n-r} + A_{n-r}S^{n-r-1} + \dots + A_2S + A_1 ,$$

r poles are required to be cancelled namely

$$(S - \lambda_{n-r+1})(\dots)(S - \lambda_n) .$$

The singular surface is found to be a (n-r)th order hyperplane described by the simultaneous solution of

$$\underline{N}(\lambda_{n-r+1}) = 0, \dots, \quad \text{and} \quad \overline{N}(\lambda_n) = 0 ,$$

where the $\overline{N}(\lambda)$ are simply calculated in the manner shown above. In this case, it is clear that the singular surface is given by

$$X_{n-r+1} + A_{n-r}X_{n-r} + \dots + A_2X_2 + A_1X_1 = 0 .$$

Once again u_s and $X_1(t)$ are completely independent of the cancelled poles; therefore, the undesired poles may be arbitrarily placed in the S-plane.

Example 3-1. All the free-time singular trajectories with their corresponding singular controls are to be determined for the third order linear LOP whose quadratic cost functional is given by

$$J[u] = \frac{1}{2}\int_0^T (X_1^2 + X_2^2 + X_3^2)\, dt \ .$$

The characteristic polynomial for X_1, u_s or $\bar{u}$ is found from (3-17) to be

$$S^4 - S^2 + 1 = 0$$

Therefore, the admissible set of singular eigenvalues is

$$\pm\left(\frac{\sqrt{3}}{2} + i\,\frac{1}{2}\right),\ \pm\left(-\frac{\sqrt{3}}{2} - i\,\frac{1}{2}\right) \ .$$

Since all the admissible eigenvalues are complex, they must be selected in conjugate pairs to insure that $\bar{u}$ will be real. Consequently, 1st and 3rd order singular trajectories do not exist. There are only two choices for the desired sets of eigenvalues, namely

$$-\frac{\sqrt{3}}{2} + i\,\frac{1}{2}\ ,\quad -\frac{\sqrt{3}}{2} - i\,\frac{1}{2}\ , \qquad (3\text{-}68)$$

and

$$+\frac{\sqrt{3}}{2} + i\,\frac{1}{2}\ ,\quad +\frac{\sqrt{3}}{2} - i\,\frac{1}{2}\ . \qquad (3\text{-}69)$$

Since neither of these sets contains a symmetric pair of eigenvalues, the singular trajectories corresponding to them are second order free-time singular arcs. The characteristic polynomials corresponding to (3-68) and (3-69) are

$$S^2 + \sqrt{3}\ S + 1 = 0\ ,$$

and

$$S^2 - \sqrt{3}\ S + 1 = 0\ ,$$

respectively. The system being considered is third order; therefore, the singular solution of $X_1(s)$ must contain one undesirable pole. Let the pole to be cancelled be arbitrarily placed at the origin of the state space, then the singular controls corresponding to (3-68) and (3-69) are

$$\bar{u}_1 = -\sqrt{3}\ X_3 - X_2\ ,$$

and

$$\bar{u}_2 = \sqrt{3}\ X_3 - X_2\ ,$$

where

$$u_{si} = -\underline{a} \cdot \underline{X} + \bar{u}_i$$

When the initial state of the singular arc is such that the pole at the origin is cancelled ($N_i(0) = 0$), the resulting singular trajectory is given by

$$\bar{N}_1(0) = X_3 + \sqrt{3}\,X_2 + X_1 = 0 \quad \text{if } u_{s1} \text{ is applied,}$$

or

$$\bar{N}_2(0) = X_3 - \sqrt{3}\ X_2 + X_1 = 0 \quad \text{if } u_{s2} \text{ is applied .}$$

Example 3-2. Consider a fourth order system described by (2-5) with the following quadratic cost functional

$$J[u] = \frac{1}{2}\int_0^T (36X_1^2 + 49X_2^2 + 14X_3^2 + X_4^2)\, dt \ .$$

All the free-time singular arcs are now determined.

The singular characteristic polynomial is given by (3-17) as

$$S^6 - 14\, S^4 + 49S^2 - 36 = 0 \ . \tag{3-70}$$

Solving (3-70), the admissible set of singular eigenvalues is found to be

$$\left\{\pm 1, \ \pm 2, \ \pm 3\right\} . \tag{3-71}$$

Since the singular state equation is fourth order, four eigenvalues are to be selected from (3-71). There are

$$\frac{6!}{4!\ 2!} = 15$$

ways of selecting a set of four eigenvalues from (3-71). Twelve of these distinct sets contain one symmetric pair, while three contain two symmetric pairs of eigenvalues. The singular trajectories and controls are now determined for two particular sets.

1. The following set is chosen

$$\left\{-1, \ -2, \ -3, \ +1\right\} \tag{3-72}$$

which corresponds to the following characteristic polynomial

$$(S+1)(S+2)(S+3)(S-1) = S^4 + 5S^3 + 5S^2 - 5S - 6 =$$
$$(S^3 + 6S^2 + 11S + 6)(S-1) = (S+1)(S^3 + 4S^2 + S - 6) \ .$$

The effective control is then

$$\bar{u} = -5X_4 - 5X_3 + 5X_2 + 6X_1 ,$$

which defines the singular control, i.e.

$$u_s = - \underline{a} \cdot \underline{X} + \bar{u} .$$

The singular trajectory corresponding to (3-72) satisfies Theorem II as a free-time singular arc only if $(S+1)$ or $(S-1)$ is cancelled by the initial state of the singular arc. Therefore, the third order free-time singular arcs are given by

$$\bar{N}(+1) = X_4 + 6X_3 + 11X_2 + 6X_1 = 0$$

if $(S-1)$ is cancelled, and

$$\bar{N}(-1) = X_4 + 4X_3 + X_2 - 6X_1 = 0$$

if $(S+1)$ is cancelled. Second and first order free-time singular arcs also exist. For example, if one additional pole is cancelled, say $(S+2)$, then the resulting singular arc is second order and is given by the intersection of

$$\bar{N}(-2) = 0 \quad \text{and} \quad \bar{N}(1) = 0 ,$$

or,

$$\bar{N}(-2) = 0 \quad \text{and} \quad \bar{N}(-1) = 0 .$$

The first order singular arcs require the cancellation of two additional poles.

2. Let the selected set be given by

$$\left\{-1,\ -2,\ +1,\ +2\right\},$$

whose characteristic polynomial is

$$(S+1)(S+2)(S-1)(S-2) = (S^2+3S+2)(S^2-3S+2)$$
$$= S^4 - 5S + 4 \ .$$

The effective control is chosen according to (3-23); therefore

$$u_s = -\underline{a} \cdot \underline{X} + \bar{u} = -\underline{a} \cdot \underline{X} + (5X_3 - 4X_1) \ .$$

Theorem II requires one of each pair of symmetric poles to be cancelled. Let (S-1) and (S-2) be the poles to be cancelled, then the second order free-time singular arc is given by the intersection of the hyperplanes

$$\bar{N}(1) = X_4 + X_3 - 4X_2 - 4X_1 = 0 , \tag{3-73}$$

and

$$\bar{N}(2) = X_4 + 2X_3 - X_2 - 2X_1 = 0 \ . \tag{3-74}$$

Eliminating X_4 from (3-73) and (3-74) yields the equation of the singular surface, namely

$$X_3 + 3X_2 + 2X_1 = 0 \ . \tag{3-75}$$

Note that the equation of the singular surface can be more easily determined by replacing S^i by X_{i+1} in the product of the uncancelled poles, i.e.,

$$\left[(S+1)(S+2)\right]_{S^i \leftarrow X_{i+1}} = \left[S^3 + 3S + 2\right]_{S^i \leftarrow X_{i+1}} = X_3 + 3X_2 + 2X_1 = 0$$

As before, cancellation of additional poles results in reduced order singular arcs.

The Fixed-Time Singular Arc

The optimal control problem in which the terminal time is specified *a priori* is termed the fixed-time problem. The Minimum Principle requires the Hamiltonian to be a non-zero constant along an optimal trajectory of the fixed-time problem. For this reason, those singular trajectories for which $H_0 \neq 0$ are referred to as fixed-time singular arcs.

In a preceding section, the constant value of the Hamiltonian, H_0, was shown to consist of up to n/2 terms, each term being a function of the initial state of the singular arc and resulting from the presence of a pair of symmetric eigenvalues in the singular solution.

The admissible set of singular eigenvalues for the rth order singular problem contains (n-r) pairs of symmetric eigenvalues. Therefore, every set of n eigenvalues selected contains at least r pairs of symmetric eigenvalues. Assume the selected set contains r pairs of symmetric eigenvalues. Then, from Theorem I, H_0 is given by

$$H_0 = \sum_{i=1}^{r} \bar{K}_i \, N(\lambda_i) N(-\lambda_i) \ ,$$

where the $\bar{K}_i$ are non-zero constants dependent solely upon the selected eigenvalues, and the $N(\lambda)$ are functions of $\underline{X}_0$, the initial singular state. Clearly, H_0 will be non-zero if and only if there exist at least one set, $\{N(\lambda_i),\ N(-\lambda_1)\}$, such that

$$N(\lambda_i) \neq 0 \quad \text{and} \quad N(-\lambda_i) \neq 0$$

for some $i \in \{1,\ldots,r\}$. This means that the fixed-time singular arcs require the presence of at least one pair of symmetric eigenvalues in the singular solution. This result may be summarized in the following theorem.

Theorem III. The singular trajectories given by Theorem I satisfy the Minimum Principle in the fixed-time case if and only if

(1) The eigenvalues of $X_1(t)$ belong to the admissible set, and

(2) $X_1(t)$ contains at least one pair of symmetric eigenvalues.

In order to analyze the structure of the fixed-time singular arcs, consider an nth order singular arc dependent upon only one pair of symmetric eigenvalues. Assume its characteristic polynomial is given by

$$(S-\lambda_1)(\ldots)(S-\lambda_{n-1})(S+\lambda_1) = (S-\lambda_1)(S^{n-1}+c_{n-1}S^{n-2}+\ldots+c_1) \quad (3\text{-}76)$$
$$= (S^{n-1}+d_{n-1}S^{n-2}+\ldots+d_1)(S+\lambda_1).$$

The initial state of the singular arc must not cancel either $(S-\lambda_1)$ or $(S+\lambda_1)$ in order for Theorem III to be satisfied. The equation of this fixed-time singular trajectory is given by Theorem I as

$$Q_n\bar{N}(\lambda_1)\ \bar{N}(-\lambda_1) = -2H_0 \ ,$$

or,

$$Q_n[X_n + c_{n-1}X_{n-1}+\ldots+c_1X_1][X_{n-1}+d_{n-1}X_{n-1}+\ldots+d_1X_1] = -2H_0 \quad (3\text{-}77)$$

Equation (3-77) is of a hyperbolic nature, which implies it must possess asymptotes. Since these fixed-time trajectories may pass through every point in the state space except those points lying on the hyperplanes defined by $\bar{N}(\lambda_1) = 0$ and $\bar{N}(-\lambda_1) = 0$, the asymptotes for (3-77) are simply

$$\bar{N}(\lambda_1) = X_n + c_{n-1}X_{n-1} + \ldots + c_1X_1 = 0 ,$$

and

$$\bar{N}(-\lambda_1) = X_n + d_{n-1}X_{n-1} + \ldots + d_1X_1 = 0 ,$$

which are the (n-1)st order free-time singular surfaces corresponding to the selected set of eigenvalues. It can be concluded that the fixed-time arcs are asymptotic to the free-time singular arcs.

Reduced order fixed-time arcs also exist. Consider the characteristic polynomial (3-76). The order of the fixed-time arc is reduced by one for every non-symmetric pole that is cancelled. For example, if $(S - \lambda_2)$ is cancelled, then the (n-1)st order fixed-time arc is given by the intersection of (3-77) and $\bar{N}(\lambda_2) = 0$. In this case (3-77) may be rewritten by solving $N(\lambda_2) = 0$ for one of the X_i and eliminating X_i from $\bar{N}(\lambda_1)$ and $\bar{N}(-\lambda_1)$. Clearly, fixed-time arcs of order two through n exist, depending on the number of non-symmetric poles cancelled.

<u>Example 3-3</u>. Consider the LOP described in Example 3-2. The fixed-time singular arcs corresponding to (3-72) are to be determined.

Since the selected set of eigenvalues contains a symmetric set, the resulting fourth order singular arc satisfies Theorem III. The singular characteristic polynomial is given by

$$\begin{aligned}(S+1)(S+2)(S+3)(S-1) &= (S^3+6S^2+11S+6)(S-1) \\ &= (S+1)(S^3+4S^2+S-6) .\end{aligned}$$

Since $Q_4 = 1$, the fourth order fixed time arc is given by Theorem I as

$$\bar{N}(1)\ \bar{N}(-1) = -2H_0$$

or,

$$[X_4 + 6X_3 + 11X_2 + 6X_1][X_4 + 4X_3 + X_2 - 6X_1] = -2H_0 , \qquad (3\text{-}78)$$

where H_0 is assumed to be non-zero. If one pole is cancelled, say (S + 2), then the resulting singular arc is of order three and is given by the intersection of (3-78) and

$$\bar{N}(-2) = X_4 + 3X_3 - X_2 - 3X_1 = 0 ,$$

i.e.,

$$3[X_3 + 4X_2 + 3X_1][X_3 + 2X_2 - 3X_1] = - 2H_0 .$$

Similarly, if the remaining non-symmetric pole, (S + 3), is cancelled, the second order fixed-time singular arc is given by

$$24[X_2 + X_1][X_2 - X_1] = -2H_0 .$$

Clearly, first order fixed-time singular arc do not exist.

Summary

In this chapter, the linear LOP with the quadratic cost functional was considered. The singularity of this problem was established by demonstrating that there do exist controls other than the bang-bang control for which the Minimum Principle can be satisfied.

All the admissible singular controls along with their corresponding singular trajectories were derived as a function of the selected set of singular eigenvalues. Since all the controls and trajectories that are candidates for optimality have been found, the optimal solution can now be determined in the manner described by Chapter I.

CHAPTER IV

SINGULAR ARCS IN NONLINEAR, TIME-VARYING, AND UNCONTROLLABLE SYSTEMS

In this chapter, the singularity of nonlinear, time-varying, and uncontrollable systems is investigated. The results obtained in Chapter III are shown to apply to certain classes of nonlinear and time-varying LOPs with quadratic cost functionals. The general nonlinear LOP is considered for which only a method of attack can be outlined. Finally, the uncontrollable LOP is discussed in relation to the singular problem.

Linear Time-Varying Systems

Consider a linear time-varying system described by the following nth order ordinary differential equation with time-varying coefficients.

$$\overset{n}{X}(t) - a_n(t)\overset{n-1}{X}(t) - \dots - a_2(t)\dot{X}(t) - a_1(t)X(t) = u(t) , \qquad (4\text{-}1)$$

where $X(t)$ is a scalar function representing the state of the system and $u(t)$ is the control or forcing function. (4-1) can be transformed into a set of n first order ordinary differential equations by letting

$$X_1 = X ,$$

and

$$\overset{i}{X}_1 = X_{i+1}$$

for $i = 1,\dots,n-1$. The phase variable form of (4-1) is then

$$\dot{X}_1 = X_2$$

$$\vdots$$

$$\dot{X}_{n-1} = X_n$$

$$\dot{X}_n = a_1(t)X_1 + \ldots + a_n(t)X_n + u = \underline{a}(t) \cdot \underline{X} + u$$

Find the control, u^*, that will transfer the system from $\underline{X}(0)$ to $\underline{X}_F$ and minimize $J[u]$ subject to system constraint (4-2) and the control constraint

$$|u| \leq 1 .$$

Let

$$J[u] = \int_0^T f(\underline{X})\, dt ,$$

where T may be fixed or free. Letting

$$X_{n+1}(t) = t ,$$

the Hamiltonian becomes

$$H = f(\underline{X}) + \sum_{i=1}^{n-1} P_i X_{i+1} + P_n \sum_{i=1}^{n} a_i(X_{n+1})X_i + P_n u , \qquad (4\text{-}3)$$

from which the adjoint equations are calculated to be

$$\dot{P}_0 = 0 = P_0(t) = +1$$

$$\dot{P}_1 = -\frac{\partial f}{\partial X_1} - a_1(X_{n+1})P_n$$

$$\dot{P}_2 = -\frac{\partial f}{\partial X_2} - a_2(X_{n+1})P_n - P_1 \qquad (4\text{-}4)$$

$$\vdots$$

$$\dot{P}_n = -\frac{\partial f}{\partial X_n} - a_n(X_{n+1})P_n - P_{n-1}$$

$$\dot{P}_{n+1} = P_n \sum_{i=1}^{n} \frac{\partial a_i}{\partial X_{n+1}} X_i \quad ,$$

where $P_{n+1}(T) = 0$ when T is free. The singular solutions exist only if $P_n(t) = 0$ over some measurable time interval. Therefore setting P_n and $\dot{P}_n = 0$ in (4-4) the singular adjoint equations are found to be

$$\begin{aligned}
\dot{P}_1 &= -\frac{\partial f}{\partial X_1} \\
\dot{P}_2 &= -\frac{\partial f}{\partial X_2} - P_1 \\
&\vdots \\
\dot{P}_n &= -\frac{\partial f}{\partial X_n} - P_{n-1} \\
\dot{P}_{n+1} &= 0 \Rightarrow P_{n+1}(t) = \text{constant} \ .
\end{aligned} \tag{4-5}$$

Singular solutions exist if there exist a system trajectory and a non-zero adjoint vector satisfying the Minimum Principle. Inspection of (4-5) indicates that ∇f must be non-zero for $\underline{P}$ to be non-zero; therefore the time optimal problem ($f = 1$) is non-singular.

When $f(\underline{X})$ is chosen to be a quadratic function the singular adjoint equations admit the possibility of singular solutions. Let

$$f(\underline{X}) = \frac{1}{2} \sum_{i=1}^{n} Q_i X_i^2 \ ,$$

and

$$\bar{u} = u_s + \sum_{i=1}^{n} a_i(X_{n+1})X_i \ ,$$

then the singular state and adjoint equations become identical to those of the linear stationary case discussed in the preceding chapter. Therefore the effective control, $\bar{u}$, is given by (3-20) as

$$\bar{u} = \sum_{i=1}^{n} b_i X_i \ ,$$

and

$$u_s = \sum_{i=1}^{n} (b_i - a_i(X_{n+1}))X_i \ .$$

The time-varying system is seen to become stationary under the application of the singular control due to the feedback cancellation property of u_s. This problem therefore contains all the fixed- and free-time singular trajectories discussed in Chapter III for the linear stationary quadratic problem.

Nonlinear Systems

Many of the examples of singular problems found in the literature are nonlinear and stationary. At present there is no general method of solution that will yield all the singular controls and trajectories for the nth order nonlinear singular problem.

This problem is formulated in the following manner. Minimize

$$J[u] = \sum_{i=0}^{n} c_i X_i(T) ,$$

where T may be fixed or free, subject to the control constraint

$$|u| \leq 1 ,$$

and the system constraint

$$\dot{X}_i = f_i(\underline{X}) + ug_i(\underline{X}) , \tag{4-6}$$

for i = 0,...,n, such that $\underline{X}$ is transferred from $\underline{X}(0)$ to $\underline{X}_F$. (4-6) is assumed to incorporate all additional differential constraints contained in the problem. State variable constraints are not considered.

The Hamiltonian is given by

$$H = \sum_{i=0}^{n} P_i f_i(\underline{X}) + u \sum_{i=0}^{n} P_i g_i(\underline{X}) ,$$

from which the switching function is found to be

$$\zeta = \frac{\partial H}{\partial u} = \sum_{i=0}^{n} P_i g_i(\underline{X}) .$$

If an admissible singular control and singular trajectory exist, then there must exist and adjoint vector $\underline{P}$ corresponding to $\underline{X}_s$ and u_s such that

$$\zeta = \sum_{i=0}^{n} P_i g_i(X_s) \equiv 0$$

during the period that u_s is applied, otherwise the problem is normal and the optimal solution is bang-bang.

Following the procedure developed by Johnson and Gibson (2), ζ and its derivatives are set equal to zero which yields the following set of equations.

$$\zeta = \sum_{i=0}^{n} P_i g_i(\underline{X}) = 0$$

$$\dot{\zeta} = \frac{d}{dt} \sum_{i=0}^{n} P_i g_i(\underline{X}) = 0 \qquad (4\text{-}7)$$

$$\vdots$$

$$\overset{i}{\zeta} = \frac{d^i}{dt^i} \sum_{i=0}^{n} P_i g_i(\underline{X}) = 0$$

where i is the first derivative of ζ to contain the control term, u_s. u_s is now solved for as a function of $\underline{X}$ and $\underline{P}$ from the last equation. Since H must be a constant, the following relation must also hold.

$$H\Big|_{\zeta=0} \sum_{i=0}^{n} P_i f_i = H_0 \quad \text{(a constant)} . \qquad (4\text{-}8)$$

Therefore, there are (i+1) equations in $\underline{X}$ and $\underline{P}$. If $\underline{P}$ can be eliminated from any one of these equations, the resulting relation in $\underline{X}$ represents the singular trajectories. Also, it is desired to eliminate $\underline{P}$ from the expression for u_s so that u_s can be realized as a feedback controller. Unfortunately this procedure for defining the singular arcs and controls (as a function of $\underline{X}$ alone) is normally only possible when $(i+1) \geq n$. Since i is typically a small number (normally 2), this method is only applicable to most second and third order systems. Two simple examples are now considered to demonstrate the technique as well as some of the difficulties encountered.

Example 4-1

Consider the sounding rocket problem solved by Johnson and Gibson (2) which is formulated in the following manner. Maximize the terminal altitude of the rocket, $X_1(T)$, subject to the system constraint

$$\dot{X}_1 = X_2 \qquad X_1(0) = 0$$

$$\dot{X}_2 = -K X_2^2 - g + u$$

$$\dot{X}_3 = u \qquad X_3(T) = b,\ T \text{ free} ,$$

and the control constraint

$$B \leq u \leq A .$$

For this problem ζ and its first i derivatives are given by

$$\zeta = P_2 + P_3 = 0$$

$$\dot{\zeta} = \dot{P}_2 + \dot{P}_3 = (-P_1 + 2KP_2X_2) + 0 = -P_1 + 2KP_2X_2 = 0$$

$$\ddot{\zeta} = -\dot{P}_1 + 2K\dot{P}_2X_2 + 2KP_2\dot{X}_2$$

$$= 2KX_2(-P_1 + 2KX_2P_2) + 2KP_2(-KX_2^2 - g + u_s) = 0 .$$

Solving $\ddot{\zeta} = 0$ for u_s yields

$$u_s = \frac{X_2P_1}{P_2} - KX_2^2 + g .$$

Since T is free, the Hamiltonian is zero, and

$$H\Big|_{\zeta=0} = P_1X_2 - P_2(KX_2^2 + g) = 0 \ .$$

Combining this equation with

$$P_2 = -P_3 \ ,$$

and

$$P_1 = 2K\ P_2X_2$$

yields

$$K\ X_2^2 - g = 0 \ ,$$

or,

$$X_2 = \sqrt{\frac{g}{K}}$$

which is the equation of the singular surface. Substituting for P_1 and P_2, u_s becomes

$$u_s = 2g \ .$$

Substituting for u_s and X_2, the state equations reduce to

$$\dot{X}_1 = X_2 = \sqrt{\frac{g}{K}}$$

$$\dot{X}_2 = 0$$

$$\dot{X}_3 = 2g \ .$$

Inspection of the singular state equations indicates that the singular control has effectively cancelled out the nonlinear properties of the original system. This is similar to the feedback cancellation property realized in the linear case of Chapter III.

Example 4-2

Gibson and Johnson (3) considered a third order nonlinear phase variable system with a quadratic cost functional; however, they were unable to completely describe the singular controls and trajectories for this LOP. The results obtained by Gibson and Johnson are now presented to demonstrate the limitations of their technique.

The system equations and cost functional are given by

$$\dot{X}_1 = X_2$$

$$\dot{X}_2 = X_3$$

$$\dot{X}_3 = f(\underline{X}) + u\, g(\underline{X}) \ ,$$

and

$$J[u] = \frac{1}{2}\int_0^T (Q_1X_1^2 + Q_2X_2^2 + Q_3X_3^2)\, dt \ ,$$

where T is free and $|u| \leq 1$. The Hamiltonian is found to be

$$H = \frac{1}{2}\sum_{i=1}^{3} Q_iX_i^2 + P_1X_2 + P_2X_3 + P_3f(\underline{X}) + uP_3g(\underline{X}) \ ,$$

from which

$$\frac{\partial H}{\partial u} = P_3g(\underline{X})$$

Assume $g(\underline{X}) \not\equiv 0$ along the singular trajectory. Then the singular condition is given by $\zeta \equiv P_3 \equiv 0$. Successively differentiating P_3 yields

$$\dot{P}_3 = -Q_3X_3 - P_2 = 0$$

$$\ddot{P}_3 = -Q_3\dot{X}_3 - \dot{P}_2 = -Q_3f - Q_3g\ u_s + Q_2X_2 + P_1 = 0 .$$

Therefore if $Q_3 \neq 0$, then

$$u_s = \frac{1}{Q_3g}\left[-Q_3f + Q_2X_2 + P_1\right] .$$

Setting $H_0 = P_3 = 0$ in H yields

$$\frac{1}{2}\sum_{i=1}^{3} Q_iX_i^2 + P_1X_2 + P_2X_3 = 0 \quad ,$$

but,

$$P_2 = -Q_3X_3 \ ;$$

therefore

$$P_1 = \frac{1}{X_2}\left[\frac{1}{2}Q_3X_3^2 - \frac{1}{2}Q_2X_2^2 - \frac{1}{2}Q_1X_1^2\right] ,$$

and u_s can be rewritten as

$$u_s = -\frac{f}{g} + \frac{1}{g}\left[-\frac{Q_1}{2Q_3}\frac{X_1^2}{X_2} + \frac{Q_2}{2Q_3}X_2 + \frac{1}{2}\frac{X_3^2}{X_2}\right] . \qquad (4\text{-}9)$$

In this example there do not exist a sufficient number of relations between $\underline{X}$ and $\underline{P}$ to define the singular trajectories; however, the singular control was found and shown to possess the property of feedback cancellation. In the following section this problem will be completely solved for the nth order case using the techniques developed in Chapter III.

Nonlinear Systems in Phase Variable Form

Consider a special case of the general nonlinear system described in the preceding section which can be described by the following nth order differential equation

$$\overset{n}{X} - f(X,\dot{X},\ldots,\overset{n-1}{X}) = u\, g(X,\dot{X},\ldots,\overset{n-1}{X}) \ , \qquad (4\text{-}10)$$

where f and g are arbitrary nonlinear functions of X and its first n-1 derivatives. (4-10) can be written in its phase variable form as

$$\begin{aligned} X &= X_1 \\ \dot{X}_1 &= X_2 \\ &\vdots \\ \dot{X}_{n-1} &= X_n \\ \dot{X}_n &= f(\underline{X}) + u\, g(\underline{X}) \end{aligned} \qquad (4\text{-}11)$$

where $\underline{X}$ is the nth order phase vector. Let

$$J[u] = \int_0^T f_0(\underline{X})\, dt \ ,$$

where T may be fixed or free. Then find the control $u^* \in \Omega$ that will transfer $\underline{X}$ from $\underline{X}_0$ to $\underline{X}_F$ and minimize J[u], where

$$\Omega = \left\{ u \;\middle|\; |u| \leq 1 \right\} \ .$$

The Hamiltonian is given by

$$H = f_0 + \sum_{i=1}^{n-1} P_i X_{i+1} + P_n f(\underline{X}) + uP_n g(\underline{X}) \ ,$$

which indicates the problem is singular if and only if

$$P_n g(X) = 0$$

along an admissible system trajectory. $g(X)$ is assumed to non-zero along all possible system trajectories since the system would be completely uncontrollable if it were zero. The condition for singularity is then

$$P_n(t) \equiv 0 \tag{4-12}$$

which must be satisfied along all singular trajectories. The adjoint equations are given by

$$\begin{aligned}
\dot{P}_0 &= 0 \Rightarrow P_0(t) = +1 \\
\dot{P}_1 &= -\frac{\partial f_0}{\partial X_1} - P_n \frac{\partial f}{\partial X_1} \\
\dot{P}_2 &= -\frac{\partial f_0}{\partial X_2} - P_n \frac{\partial f}{\partial X_2} - P_1 \\
&\vdots \\
\dot{P}_n &= -\frac{\partial f_0}{\partial X_n} - P_n \frac{\partial f}{\partial X_n} - P_{n-1} .
\end{aligned} \tag{4-13}$$

Substituting the singular condition into (4-13) yields

$$\dot{P}_1 = -\frac{\partial f_0}{\partial X_1}$$

$$\begin{aligned} \dot{P}_2 &= -\frac{\partial f_0}{\partial X_2} - P_1 \\ &\vdots \\ \dot{P}_n &= -\frac{\partial f_o}{\partial X_n} - P_{n-1} = 0 \end{aligned} \tag{4-14}$$

Once again the Minimum Principle is violated if $\nabla f_0 = 0$, thereby ruling out singular time optimal arcs. Let f_0 be quadratic, i.e.,

$$f_0 = \frac{1}{2} \sum_{i=1}^{n} Q_i X_i^2 \ ;$$

therefore, $\nabla f_0 \neq 0$. Define the effective control, $\bar{u}$, as

$$\bar{u} = f(\underline{X}) + u_s \, g(X) \ ,$$

which implies

$$u_s = \frac{\bar{u} - f(\underline{X})}{g(\underline{X})} \ . \tag{4-15}$$

Once again the singular state equations and the singular adjoint equations become identical to those of the linear stationary problem considered in Chapter III. Therefore

$$\bar{u} = \sum_{i=1}^{n} b_i X_i = \underline{b} \cdot \underline{X} \ , \tag{4-16}$$

and

$$u_s = \frac{\underline{b} \cdot \underline{X} - f(\underline{X})}{g(\underline{X})} \ .$$

The singular trajectories are calculated in the same manner as before.

Example 4-3

Apply the above techniques to the problem considered in Example 4-2 and compare the results.

Let $Q_1 = 4$, $Q_2 = 5$, $Q_3 = 1$; then the admissible set of eigenvalues is

$$\left\{\pm 1, \pm 2\right\} .$$

Since Example 4-2 considered a free-time problem, the desired eigenvalues must be chosen to be

$$\left\{-1, -2\right\}, \quad \text{or} \quad \left\{+1, +2\right\}$$

both of which correspond to a free-time singular arcs. Assume the stable set of eigenvalues is chosen. The additional pole is placed at the origin for convenience. The characteristic polynomial for the singular system is then

$$(S+1)(S+2)S = (S^2 + 3S + 2)S = S^3 + 3S^2 + 2S \ .$$

The equation of the second order free-time singular arcs is given by

$$\bar{N}(0) = X_3 + 3X_2 + 2X_1 = 0 \qquad (4\text{-}17)$$

which is realized by cancelling the pole at the origin. The effective control corresponding to the selected set of eigenvalues is given by

$$\bar{u} = -3x_3 - 2X_2 \qquad (4\text{-}18)$$

which implies

$$u_s = -\frac{f}{g} + \frac{1}{g}(-3X_3 - 2X_2) \ .$$

The expression for u_s calculated by Gibson and Johnson in Examples 4-2 was given by (4-9) as

$$u_s = -\frac{f}{g} + \frac{1}{g}\left[-2\frac{X_1^2}{X_2} + \frac{5}{2}X_2 + \frac{1}{2}\frac{X_3^2}{X_2}\right]$$

which corresponds to the nonlinear effective control

$$\bar{u} = \frac{1}{2X_2}\left[X_3^2 + 5X_2^2 - 4X_1^2\right]. \qquad (4\text{-}19)$$

It was shown that the singular trajectories must be on the singular surface given by (4-17). From (4-17)

$$-4X_1^2 = -(X_3 + 3X_2)^2 = -X_3^2 - 6X_2X_3 - 9X_2^2 ,$$

which when substituted in (4-19) yields

$$\bar{u} = \frac{1}{2X_2}\left[-6X_2X_3 - 4X_2^2\right] = -3X_2 - 2X_2 \ .$$

Therefore, the expression for u_s obtained by Gibson and Johnson reduces to the much simpler result (4-18) developed in this research when the state of the system is on the singular surface.

In this example, the previously unsolved problem considered by Gibson and Johnson was considered. Application of the technique developed in the research for establishing singularity of the nth order nonlinear phase variable LOP was shown to define all the singular controls and trajectories as well as demonstrate the mechanism of pole-zero cancellation associated with the singular trajectories.

Uncontrollable Systems

Hermes (9) has extended Kalman's (8) concept of complete controllability to nonlinear systems, and has shown that all LOPs, both linear and nonlinear, are singular if they are completely uncontrollable. The converse of this statement is not necessarily true. Hermes defines an LOP to be singular if and only if the solution admits a totally singular arc, i.e., an arc satisfying the differential constraint equations, for which there exists an adjoint vector such that the Minimum Principle yields no information as to the optimality of any component of the control along the arc, i.e., the Hamiltonian becomes explicitly independent of the control along the totally singular arc.

The major problem associated with uncontrollable systems is the fact that there is no guarantee that a solution exists. This is especially true when the target set is a point in the state space. From Hermes work, it is known that uncontrollability implies singularity; however, this fact does not imply that the optimal solution contains singular arcs.

Totally singular arcs must first satisfy the necessary condition for singular extremals before they can be considered as possible optimal trajectories.

A linear second order uncontrollable system is examined for various cost functionals to determine if admissible singular arcs exist. Consider the linear system described by

$$\dot{\underline{X}} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \underline{X} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u = A\underline{X} + \underline{B}u , \tag{4-20}$$

which is uncontrollable since

$$\det \left[\underline{B} \vdots A\underline{B}\right] = \det \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 0 .$$

Let the cost functional to be minimized be

$$J[u] = \int_0^T f_0(\underline{X})\, dt ;$$

then the Hamiltonian and switching function are

$$H = f_0 + P_1X_2 + P_2(-X_1 - 2X_2) + u(P_1 - P_2) , \tag{4-21}$$

and

$$\zeta = P_1 - P_2 .$$

Therefore singular arcs are possible only if $(P_1 - P_2) \equiv 0$. The singular adjoint equations are

$$\begin{aligned} \mathring{P}_1 &= - \frac{\partial f_0}{\partial X_1} + P_2 = - \frac{\partial f_0}{\partial X_1} + P_1 \\ \mathring{P}_2 &= - \frac{\partial f_0}{\partial X_2} - P_1 + 2P_2 = - \frac{\partial f_0}{\partial X_2} + P_1 . \end{aligned} \tag{4-22}$$

Case I

Let $f_0 = 1$ (the time optimal case), then (4-21) and (4-22) become

$$H = 1 - P_1[X_1 + X_2] ,$$

and

$$\begin{aligned} \dot{P}_1 &= P_1 \\ \dot{P}_2 &= P_1 \end{aligned} \quad . \tag{4-23}$$

In this case $\overset{i}{\zeta} = (\overset{i}{P}_1 - \overset{i}{P}_2) \equiv 0$ for any i, since $\dot{P}_1 \equiv \dot{P}_2 \equiv P_1$. Therefore differentiating ζ does not produce an expression for u_s. From (4-23), one finds

$$P_1 = P_{10}\, e^{-t} ;$$

therefore, H is a constant only if

$$X_1(t) + X_2(t) = (X_1(0) + X_2(0))e^{+t} , \tag{4-24}$$

where the singular control is assumed to be applied at time $t_s = 0$. The singular state equations are given by (4-20) as

$$\dot{X}_1 = X_2 + u_s$$

$$\dot{X}_2 = -X_1 - 2X_2 - u_s ,$$

where u_s must be chosen so that X_1 and X_2 satisfy (4-24). Adding $\dot{X}_1$ and $\dot{X}_2$ yields

$$\dot{X}_1 + \dot{X}_2 = X_2 + u_s - X_1 - 2X_2 - u_s ,$$

or,

$$(\overset{\circ}{X_1 + X_2}) = -(X_1 + X_2) \ ,$$

which implies that (4-24) can be satisfied for any u_s satisfying the control constraint. Therefore, for every possible system trajectory, there exist an adjoint vector such that the Hamiltonian is independent of u, i.e., every system trajectory can be considered a singular trajectory. Since

$$(-1)^r \frac{\partial}{\partial u} \frac{d^{2r}}{dt^{2r}} (P_1 - P_2) = 0$$

for all r, the required necessary condition is not satisfied which implies the singular arcs cannot be optimal.

Case II

Let $f_0 = \frac{1}{2}(X_1^2 + X_2^2)$, then (4-21) and (4-22) become

$$H = \frac{1}{2}(X_1^2 + X_2^2) - P_1(X_1 + X_2) \ ,$$

and

$$\overset{\circ}{P}_1 = -X_1 + P_1$$

$$\overset{\circ}{P}_2 = -X_2 + P_1 \ .$$

Differentiating ζ yields

$$\overset{\circ}{\zeta} = X_2 - X_1 + P_1 - P_2 = 0 \Rightarrow X_1 = X_2$$

$$\overset{\circ\circ}{\zeta} = -2u_s - 2X_1 - 2X_2 + P_1 - P_2 = 0 \Rightarrow$$

$$u_s = -2X_1 = -2X_2 \ .$$

The singular state equations become

$$\mathring{X}_1 = X_2 + u_s = -X_2 = -X_1$$

$$\mathring{X}_2 = -X_1 - 2X_2 - u_s = -X_1 = -X_2 ,$$

or,

$$X_1(t) = X_2(t) = X_{10}e^{-t}$$

The singular adjoint vector must be chosen to insure the Hamiltonian is a constant, i.e.,

$$H\Big|_{X_1=X_2} = X_1^2 - 2P_1X_1 = X_{10}^2\, e^{-2t} - 2X_{10}e^{-t}P_1 = H_0 \text{ (constant)} .$$

Clearly,

$$P_1 = \frac{1}{2}\, X_{10}e^{-t} = \frac{1}{2}\, X_1 = \frac{1}{2}\, X_2 = P_2 ,$$

and

$$H_0 = 0$$

In this case, the necessary condition for singular extremals is satisfied, i.e.,

$$-1\, \frac{\partial}{\partial u}\, [\ddot{\zeta}] = +2 > 0 .$$

Since all the necessary conditions are satisfied, the singular arc,

$$X_1 = X_2 ,$$

is admissible as an optimal trajectory.

The preceding example shows that the uncontrollable, linear, second order LOP admits singular trajectories for both the time-optimal and quadratic cost functionals. In fact, in the time-optimal case (Case I), every control function satisfying the control constraint can be considered a singular control since there exists an adjoint vector such that the Hamiltonian is independent of the control; however, these singular controls fail to satisfy the necessary condition for singular extremals, and therefore are not candidates for optimality. Therefore, if an optimal control function exist for Case I, then it must be bang-bang.

This example points out the fact that even though the uncontrollable LOP is singular, i.e., its solution admits singular controls which satisfy the Minimum Principle, the singular controls may not be admissible candidates for optimality.

Summary

In this chapter, the singularity of the nonlinear and time-varying phase variable LOP was shown to be a function of the cost functional with the quadratic cost functional producing singularity.

In both the nonlinear and time-varying cases, the singular controls were derived and shown to exhibit the property of "feedback cancellation," i.e., the nonlinear and time-varying systems become linear and stationary under the application of the singular control. Since the singular state and adjoint equations for these systems are identical to those of the linear system considered in Chapter III, the "effective control" as well as the corresponding fixed- and free-time singular trajectories previously derived are directly applicable.

CHAPTER V

SINGULAR SOLUTIONS

In Chapter III, the Minimum Principle was used to select all the extremal controls namely, the bang-bang control and the singular controls. If an optimal control function exists, then it must be bang-bang, singular, or some combination of the extremal controls. The singular control function was derived as a function of the set of n singular eigenvalues selected from the admissible set; therefore, for each set of n eigenvalues chosen, there exists a unique singular control function. The singular trajectories corresponding to a particular singular control were derived and shown to be dense in the state space, i.e., for every initial singular condition, there exist either a fixed- or free-time singular trajectory. These singular arcs are admissible as optimal arcs in that region of the state space where u_s satisfies the control constraint.

In Chapter I, a procedure for solving the singular optimal control problem was given which requires the construction of all the possible solution trajectories. This becomes rather difficult when the order of the system is large; however, the knowledge of the position, shape, and direction of the singular arc for any initial condition is an extremely useful aid in constructing the solution trajectories.

In order to illustrate the use of singular arcs, the regulator problem is now considered.

Regulator Problem

The regulator problem is a special case of the general problem under consideration in which the target set is the origin of the state space. Consider the following problem formulation. Minimize

$$J[u] = \frac{1}{2}\int_0^T \left[\sum_{i=1}^{n} Q_i X_i^2 + Q_T\right] dt \tag{5-1}$$

subject to the system constraint

$$\dot{X}_1 = X_2, \ldots, \dot{X}_{n-1} = X_n \tag{5-2}$$

$$\dot{X}_n = f(\underline{X}) + u\, g(\underline{X}) \quad ,$$

where

$$\underline{X}(0) = \underline{X}_0 \quad \text{and} \quad \underline{X}(T) = 0 \ ,$$

and the control constraint

$$|u| \leq 1 \ .$$

Equation (5-1) may be rewritten as

$$J[u] = \frac{1}{2}\int_0^T \sum_{i=1}^{n} Q_i X_i^2 dt + \frac{1}{2} T\, Q_T \ ,$$

or,

$$J[u] = J_1[u] + \frac{1}{2} T\, Q_T$$

where $J_1[u]$ is the standard quadratic cost function in $\underline{X}$. The additional term, $\frac{1}{2}TQ_T$, serves to weight time along with the states thereby eliminating $T = \infty$ as a possible time of arrival at the target. This formulation admits three cases of interest, namely

1. $Q_T = 0$, T free
2. $Q_T = 0$, T fixed
3. $Q_T > 0$, T free .

The admissible singular arcs for Cases I and II were derived in Chapter III. The Hamiltonian for Case III differs from that of Case II by a constant, i.e.,

$$H^{III} = H^{II} + \frac{1}{2} Q_T$$

Since $H_0^{III} = 0$, the singular arcs for Case III can be written

$$H^{II} = \sum_{i=1}^{r} \overline{K}_i \, \overline{N}(\lambda_i) \, \overline{N}(-\lambda_i) = -\frac{1}{2} Q_T$$

i.e., the singular arcs for Case III are identical to the fixed-time singular arcs of Case II when

$$H_0^{II} = -\frac{1}{2} Q_T ,$$

or,

$$\sum_{i=1}^{r} \overline{K}_i N(\lambda_i) \, N(-\lambda_i) = -\frac{1}{2} Q_T .$$

Since Cases II and III both constrain the terminal time -- Case II by a fixed final time constraint and Case III by assigning cost to the final time, it is clear that Case III can be considered as an alternate formulation for the fixed-time problem (Case II) when the final time, T, is important but not critical. Case III is the preferred formulation because the number of admissible singular arcs is extremely small compared to the infinity of singular arcs admissible for Case II, thereby making its solution comparatively easy.

The complete optimal solution of a second order regulator problem is now determined following the procedure outlined in Chapter I. The second step of this procedure requires all the admissible singular controls and trajectories to be found. The completion of this step, which is absolutely essential for the solution of the problem, is now possible due to results of this research contained in Chapter III.

Example

Find the optimal control function that will transfer the state of the system from

$$\underline{X}(0) = \underline{X}_0 = \begin{bmatrix} -6 \\ 0 \end{bmatrix} \quad \text{to} \quad \underline{X}(T) = \underline{X}_F = \underline{0} \ ,$$

and minimize the cost functional

$$J[u] = \frac{1}{2}\int_0^T \left(\frac{1}{4} X_1^2 + X_2^2 + Q_T\right) dt$$

subject to the system constraint

$$\dot{X}_1 = X_2$$

$$\dot{X}_2 = u \ ,$$

and the control constraint

$$|u| \leq 1 .$$

The optimal solution is to be determined for three cases of interest, namely

1. $Q_T = 0$, T free
2. $Q_T = 0$, $T = 5.5$
3. $Q_T = .9$, T free .

Step I

Constructing the Hamiltonian yields

$$H = \frac{1}{2} Q_T + \frac{1}{8} X_1^{\ 2} + \frac{1}{2} X_2^{\ 2} + P_1 X_2 + P_2 u$$

from which the bang-bang control is found to be

$$u_B = - \text{sign} [P_2] .$$

Step II

The singularity of this LOP was established in Chapter III. The singular characteristic polynomial was given by (3-17) as

$$S^2 - \frac{1}{4} = 0$$

which admits the following set of admissible eigenvalues

$$\left\{+ \frac{1}{2} , - \frac{1}{2}\right\} \tag{5-3}$$

Since the system under consideration is second order, two eigenvalues must be selected from (5-3) for the existence of second order singular trajectories. In this case, there is only one possible choice for the singular poles, namely

$$(S + \frac{1}{2})(S - \frac{1}{2}) = S^2 - \frac{1}{4}$$

The singular control producing these poles is found to be

$$u_s = \bar{u} = \frac{1}{4} X_1 \tag{5-4}$$

which implies the singular trajectories are admissible in that region of the state space where $|X_1| \leq 4$. The singular trajectories corresponding to (5-4) are given by Theorem I as

$$Q_n \bar{N}\ (\frac{1}{2})\ \bar{N}(-\ \frac{1}{2}) = -2(H_0 - \frac{1}{2} Q_T)\ ,$$

or,

$$(X_2 + \frac{1}{2} X_1)(X_2 - \frac{1}{2} X_1) = -2(H_0 - \frac{1}{2} Q_T)\ , \tag{5-5}$$

where H_0 is a function of the initial state of the singular arc. The state space is clearly dense in singular arcs since for every initial state there corresponds an H_0 and consequently a singular trajectory. Figure 1 shows the shape and direction of these admissible singular arcs as a function of $(H_0 - \frac{1}{2} Q_T)$.

The singular arcs which are candidates as optimal trajectories in Case I, II, and III are now selected from the admissible set given by (5-5)

<u>Case I</u> $Q_T = 0$ and T is free. Since T is free, $H_o^I = 0$ and (5-5) becomes

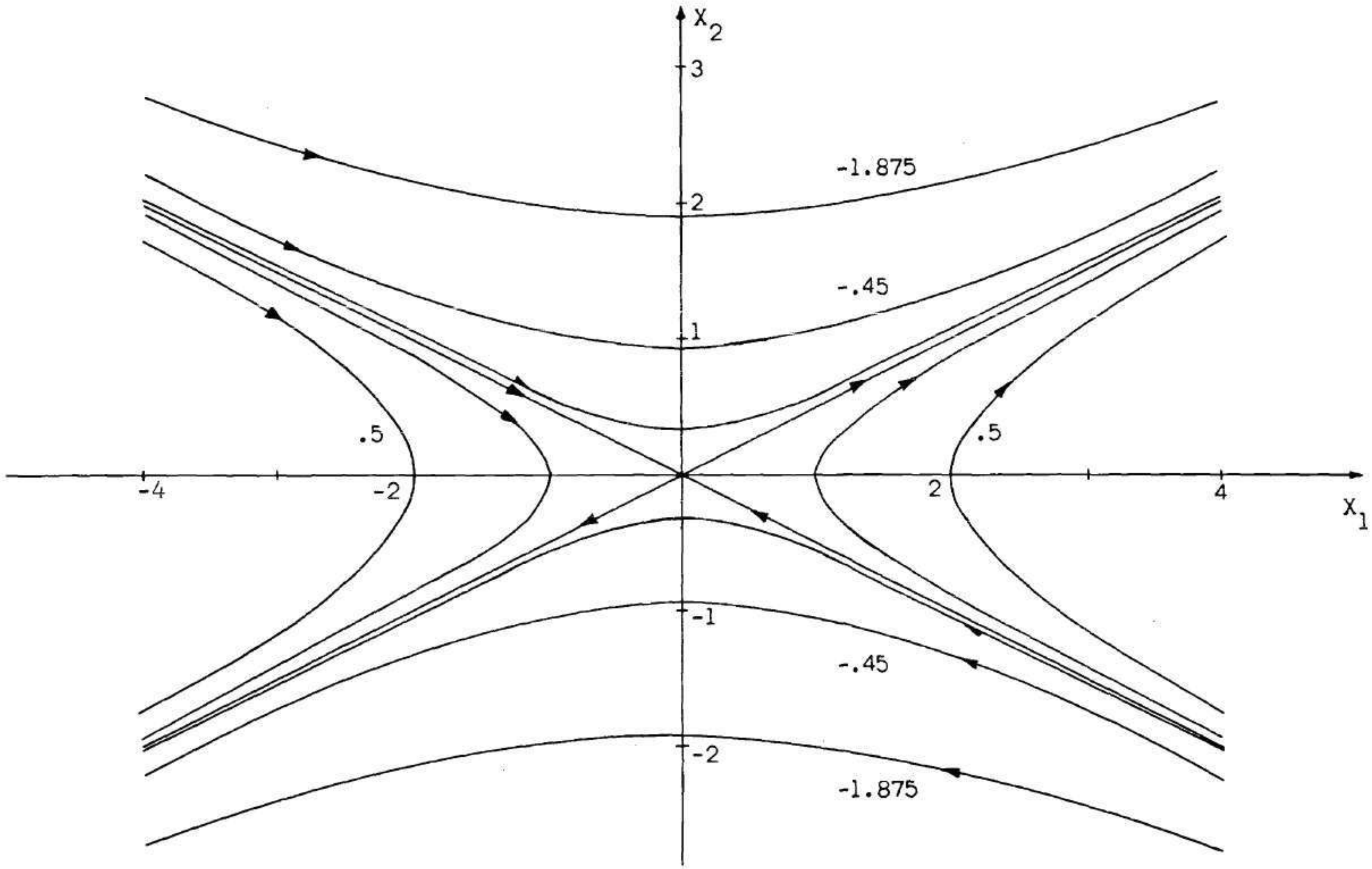

Figure 1. The Admissible Singular Arcs as a Function of $(H_0 - \frac{1}{2} Q_T)$.

$$\bar{N}(\tfrac{1}{2})\ \bar{N}(-\tfrac{1}{2}) = (X_2 + \tfrac{1}{2} X_1)(X_2 - \tfrac{1}{2} X_1) = 0$$

The selected set of eigenvalues contains a symmetric pair; therefore, one of the symmetric poles must be cancelled for Theorem II to be satisfied. The (n-1)st order hyperplanes (singular surfaces) are given by

$$X_2 + \tfrac{1}{2} X_1 = 0$$

if $(S - \frac{1}{2})$ is cancelled, and

$$X_2 - \tfrac{1}{2} X_1 = 0$$

If $(S + \frac{1}{2})$ is cancelled. Note that in this simple example, the singular surfaces are only singular arcs. These free-time singular arcs are represented in Figure 1 as the straight line trajectories.

Case II. $Q_T = 0$ and $T = 5.5$. Since T is fixed, H_0^{II} must be a non-zero constant. The admissible singular trajectories are then

$$(X_2 + \tfrac{1}{2} X_1)(X_2 - \tfrac{1}{2} X_1) = -2\ H_0^{II} \neq 0 \qquad (5\text{-}6)$$

These fixed-time singular arcs are represented by all the hyperbolas in Figure 1.

Case III $Q_T = .9$, T is free. Since T is free, $H_0^{III} = 0$; therefore, the singular arcs are given by

$$(X_2 + \tfrac{1}{2} X_1)(X_2 - \tfrac{1}{2} X_1) = .9\ .$$

This singular arc is simply one of the singular arcs described by (5-6), namely

$$H_0^{II} = -.45 \ .$$

Step III

The solution trajectories are now constructed from the set of admissible arcs for each case.

Case I. Since the admissible free-time singular arcs do not pass through the initial state of the system, the initial system subarc must be bang bang. Examination of the state equations shows that the bang-bang control $u = -1$ moves the system away from the target while $u = +1$ forces the system toward the origin; therefore, the initial control is given by $u = +1$. Inspection of Figure 2, which shows the admissible free-time singular and bang-bang arcs, indicates that there are only two solution trajectories that must be considered. They are the bang-bang solution and the composite solution which consist of the initial bang-bang subarc and the terminal singular arc. These solutions are shown in Figure 2. Evaluating the cost functional along each of these solution trajectories yields

$$J[u] \quad \text{(along the bang-bang solution)} = 13.354, \ T = 4.90 \ ,$$

and

$$J[u] \quad \text{(along the composite solution)} = 12.533, \ T = \infty \ .$$

Therefore, the optimal solution is the composite solution which contains both bang-bang and singular subarcs. Note that in order to realize the singular arc required for the optimal solution it is necessary to cancel the right-half plane pole, $(S - \frac{1}{2})$. This point will be considered in

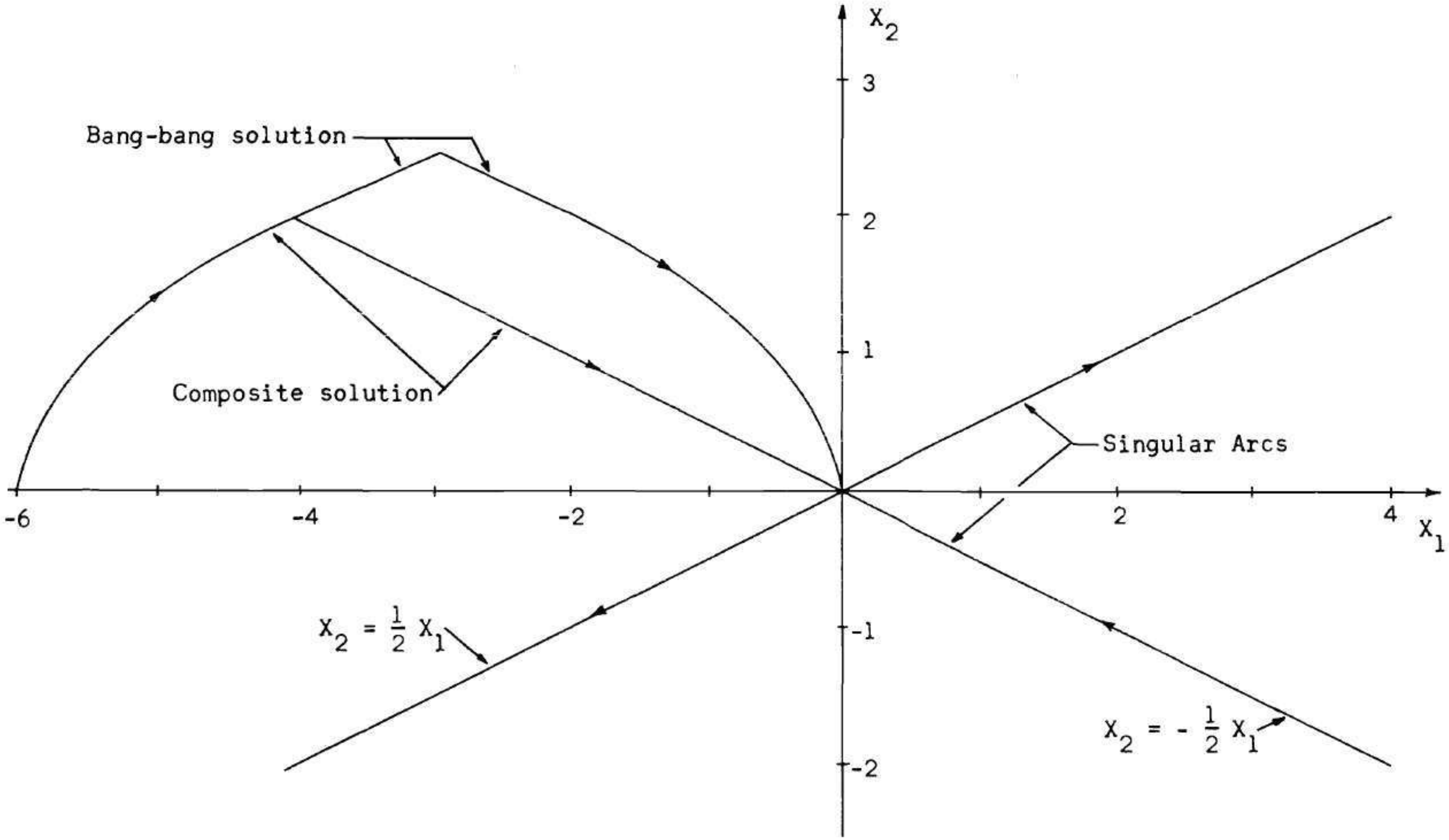

Figure 2. The Admissible Arcs and Solution Trajectories for Case I.

detail in the following section on sensitivity.

Case II. Since T > T time-optimal, an optimal solution should exist. As in Case I, the initial control must be u = + 1, since both u = -1 and u = X_1 force the system away from the target. The bang-bang solution found in Case I reaches the target in less than 5.5 seconds with a cost of 13.354. Examination of the bang-bang solutions requiring more than one switch to reach the origin are clearly more costly than the one switch solution. Inspection of Figures 1 and 2 indicates that there do exist composite solutions reaching the target in 5.5 seconds or less. These composite solutions consist of initial and terminal bang-bang arcs with an intermediate fixed-time singular arc, where the singular arcs of interest lie between the free-time singular arc and the bang-bang arc in the second quadrant. Calculations show that the minimizing composite solution reaches the target at T = 5.5 seconds and uses the intermediate singular arc characterized by $H_0^{II} = -.45$,

$$(X_2 + \frac{1}{2} X_1)(X_2 - \frac{1}{2} X_1) = -2H_0^{II} = .9 \ .$$

Evaluating the cost functional along this composite solution yields J[u] = 12.875. Therefore, the composite solution is the optimal solution. The bang-bang solution and optimal composite solution are shown in Figure 3.

Case III. The only admissible singular arc for this case is given by

$$(X_2 + \frac{1}{2} X_1)(X_2 - \frac{1}{2} X_1) = .9$$

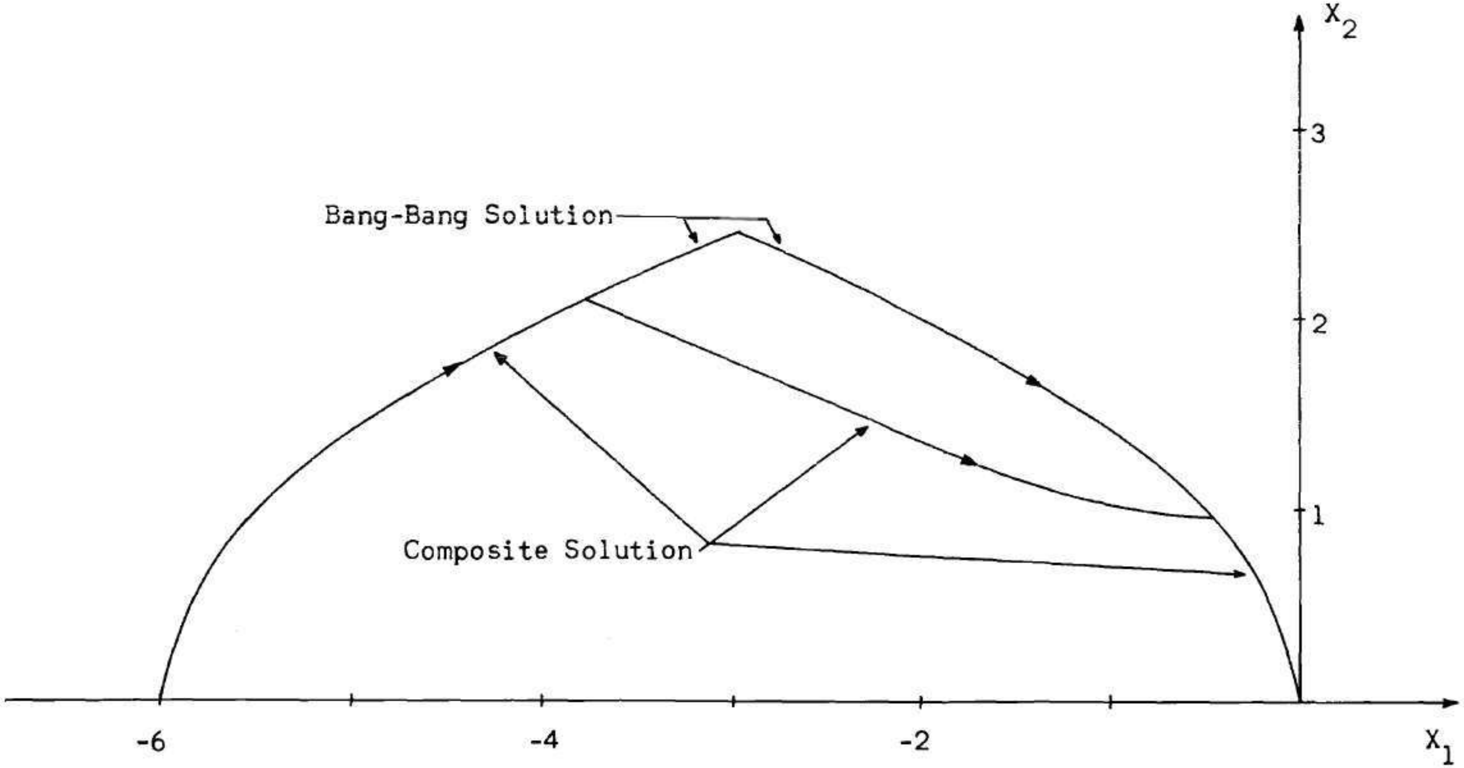

Figure 3. The Bang-Bang and Optimal Composite Solution for Case II.

which is identical to the singular arc employed in the optimal composite solution of Case II. Evaluating the cost functional of Case III along the composite solution and bang-bang solution shows that the composite solution is optimal. Clearly, the formulation of Case III is much easier to solve than Case II because the final time is free and the number of admissible singular arcs is very small.

Sensitivity Analysis

In Chapters III and IV, the singular control function was derived and shown to possess the property of feedback cancellation. In addition, the realization of the free-time as well as the reduced order fixed-time singular arcs was shown to require pole-zero cancellation due to the initial state of the singular trajectory. These two characteristics are the primary sources of error associated with the singular trajectories. The sensitivity of the singular solution to each of these errors is now determined separately.

Control Errors

For the system described by

$$\begin{aligned} \dot{X}_1 &= X_2 \\ &\vdots \\ \dot{X}_{n-1} &= X_n \\ \dot{X}_n &= f(\underline{X}) + u\, g(\underline{X}) \,, \end{aligned} \tag{5-7}$$

the singular control was found to be

$$u_s = \frac{\bar{u} - f(\underline{X})}{g(\underline{X})}$$

where $\bar{u}$ is linear in $\underline{X}$. Since a general analysis of a nonlinear system is impossible in closed form, only the linear case is considered; i.e., let

$$g(\underline{X}) = 1 \quad \text{and} \quad f(X) = \underline{a} \cdot \underline{X} \ .$$

Assume the applied control is perturbed as follows

$$u = u_s + h = \bar{u} - \underline{a} \cdot \underline{X} + h \ ,$$

where h is the admissible perturbation, then the singular state equations can be written as

$$\begin{aligned} \dot{X}_1 &= X_2 \\ &\vdots \\ \dot{X}_{n-1} &= X_n \\ \dot{X}_n &= \bar{u} + h = \underline{b} \cdot \underline{X} + h \ , \end{aligned}$$

or,

$$\dot{\underline{X}} = A\underline{X} + \underline{B}h \ . \qquad (5\text{-}8)$$

Since (5-8) is linear, its solution for any linear h can be written

$$\underline{X}(t,t_s;\, h) = \Phi(t)\Phi^{-1}(t_s)\underline{X}_0 + \int_{t_s}^{t} \Phi(t)\Phi^{-1}(\tau)\underline{B}h(\tau)d\tau \qquad (5\text{-}9)$$

where Φ is the fundamental solution matrix of (5-8). Define the sensitivity function of $\underline{X}$ with respect to h as

$$\Delta\underline{X}(h) = \underline{X}(t,t_s;\, h) - \underline{X}(t,t_s;\, h=0) \ .$$

Upon substituting, the sensitivity function becomes

$$\Delta\underline{X}(h) = \int_{t_s}^{t} \Phi(t)\ \Phi^{-1}(\tau)\ \underline{B}h(\tau)\ d\tau\ . \tag{5-10}$$

Since $\Delta\underline{X}(h)$ is linear in h, $\Delta\underline{X}(h)$ is actually the variation of $\underline{X}$ corresponding to h. Equation (5-10) can be applied to give a quantitative measure of the systems sensitivity to perturbations in the singular control.

<u>Pole-zero Cancellation Errors</u>

The pole-zero cancellation required by singular trajectories was shown to be strictly a function of the initial singular condition. The error resulting from failure to exactly an undesired pole is clearly a function of the undesired pole's position in the S-plane. This is readily seen by considering a singular solution which requires the cancellation of $(S - \lambda_n)$ in $X_1(S)$, i.e.,

$$X_1(t) = \sum_{i=1}^{n-1} A_i e^{\lambda_i t} + A_n e^{\lambda_n t}\ .$$

The error associated with each of the state variables resulting from failure to cancel $(S - \lambda_n)$ is given by

$$\Delta X_i(t) = A_n \lambda_n^{i-1}\ e^{\lambda_n t} \tag{5-11}$$

where A_n is proportional to the difference vector between the desired value of $\underline{X}_0$ and the actual value of $\underline{X}_0$. Inspection of (5-11) indicates that the resulting error increases with time when $\lambda_n > 0$ and decreases when $\lambda_n < 0$.

In Chapter III, it was shown that a cancelled pole in no way affects the singular solution; therefore, the poles to be cancelled may be arbitrarily placed in the S-plane by proper choice of $\bar{u}$. Clearly, if pole-zero cancellation errors are to be minimized, the undesired poles should be placed well into the left half of the S-plane.

A quantitative expression for the pole-zero cancellation error, ε_p, is determined in the following manner. Let

$$\varepsilon_p = \underline{X}(t,t_s; \underline{X}_0 + \underline{K}) - \underline{X}(t,t_s; \underline{X}_0)$$

where $\underline{K}$ is the difference vector between the desired and actual values of $\underline{X}_0$, then from (5-9) ε_p becomes

$$\varepsilon_p = \Phi(t)\ \Phi^{-1}(t_s)\ \underline{K}\ .$$

In most physical situations, pole-zero cancellation will be practically impossible; however, proper placement of the poles to be cancelled will insure that the actual system trajectory approaches the desired singular trajectories rapidly.

CHAPTER VI

A SUBOPTIMAL SOLUTION

In this chapter, a technique for normalizing the singular LOP is developed which leads to a suboptimal solution of the singular problem.

Within the framework of the linear optimization problem (LOP), there are many problems for which neither normality nor singularity can be established, i.e., the singular state and adjoint equations indicate that singular trajectories may exist, but there does not exist enough information to define them. The general nonlinear LOP considered in Chapter IV is such a problem. In these cases, the bang-bang solution is the only solution trajectory that can be found. Since the optimality of the bang-bang solution is unknown, a suboptimal technique which is known to produce solutions close to the optimal is extremely desirable as an instrument for determining the optimality of the bang-bang solution, i.e., the cost functional can be evaluated along both the suboptimal and bang-bang solutions to determine which solution is closest to the optimal. When the bang-bang solution is the better of the two, it *may* be the true optimal solution; however, if the suboptimal solution is the better solution, then the true optimal solution must contain one or more of the singular arcs which could not be determined. In this case, the suboptimal solution must be accepted as the best solution attainable.

The need for a suboptimal solution also arises in many problems when the optimal control cannot be physically implemented. For example, the optimal solution for the phase variable nonlinear system considered in Chapter IV contains a singular subarc which requires the singular control to completely cancel the nonlinear dynamics of the system. In such cases a more practical solution which is close to the optimal is desired.

A general technique for determining a suboptimal solution of the singular or indeterminate LOP is now developed.

Consider the following LOP. Minimize

$$J[u] = \Phi[\underline{X}(T)]$$

subject the differential constraints

$$\dot{\underline{X}} = \underline{f}(\underline{X}, t) + u\underline{g}(\underline{X}, t) \ , \tag{6-1}$$

and the control constraint, $|u| \leq 1$. When the Minimum Principle is applied, the minimizing control is found to be

$$u = - \text{ sign } [\underline{P} \cdot \underline{g}] \ . \tag{6-2}$$

The singular controls arise from the fact that the above control is not well defined for all $\underline{X}$ and $\underline{P}$ satisfying the canonical equations, i.e., (6-2) is not defined when the argument of the sign function is identically zero. Therefore, if the original problem could be modified so that the control minimizing the Hamiltonian is well defined for all possible $\underline{X}$ and $\underline{P}$, then the resulting problem would be normal.

The proposed technique for normalizing the singular or indeterminate LOP requires augmenting the system equations so the new Hamiltonian is a function of u^2, namely

$$\overline{H} = H + \alpha u^2 = [\underline{P} \cdot \underline{f} + u\underline{P} \cdot \underline{g}] + \alpha u^2 \tag{6-3}$$

where α is a positive constant. This is accomplished by augmenting the system equations by

$$\dot{X}_0 = \alpha\, u^2 \ ,$$

and adding the term $X_0(T)$ to the cost functional, i.e.,

$$\overline{J}[u] = J[u] + \int_0^T \alpha u^2 dt = \Phi[\underline{X}(T)] + X_0(T) \ . \tag{6-4}$$

When the original cost functional contains an integral term, the above procedure is equivalent to adding the term αu^2 to it. Minimizing H with respect to u yields

$$u = -\frac{1}{2\alpha}\,(\underline{P} \cdot \underline{g}) \tag{6-5}$$

which is clearly well defined for all possible $\underline{X}$ and $\underline{P}$; therefore, the modified problem is normal and may be solved directly by standard optimization procedures such as Bellman's Dynamic Programming or the Pontryagin Minimum Principle. Combining equation (6-5) with the control constraint yields the optimal control function for the normalized problem, namely

$$u^* = \left\{ \begin{array}{lll} +1 & \text{when} & -\frac{1}{2\alpha}\,(\underline{P} \cdot \underline{g}) > 1 \\ -\frac{1}{2\alpha}(\underline{P} \cdot \underline{g}) & \text{when} & \left| -\frac{1}{2\alpha}\,(\underline{P} \cdot \underline{g}) \right| \leq 1 \\ -1 & \text{when} & -\frac{1}{2\alpha}\,(\underline{P} \cdot g) < -1 \end{array} \right\} \tag{6-6}$$

If α is chosen small enough such that

$$X_0(T) << \Phi[\underline{X}(T)] ,$$

then the normalized solution should be close to the optimal solution of the original LOP.

A second order example, whose optimal solution contains a free-time singular arc, is now considered to demonstrate the technique developed. It is shown that the resulting suboptimal solution compares very favorably with the optimal solution.

<u>Example</u>

Find the control function u, where $|u| < 1$, that will transfer the state of the system $\underline{X}$ from

$$\begin{bmatrix} -1.5 \\ 0 \end{bmatrix} \text{ to } \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and minimize the cost functional

$$J[u] = \frac{1}{2}\int_0^T (X_1^2 + X_2^2)\, dt \quad (T \text{ free})$$

subject to the system constraint

$$\dot{X}_1 = X_2$$

$$\dot{X}_2 = u$$

The optimal solution trajectory which contains an initial bang-bang arc and a terminal free-time singular arc is shown in Figure 4 along with the bang-bang solution. The cost associated with each of these solutions is given by

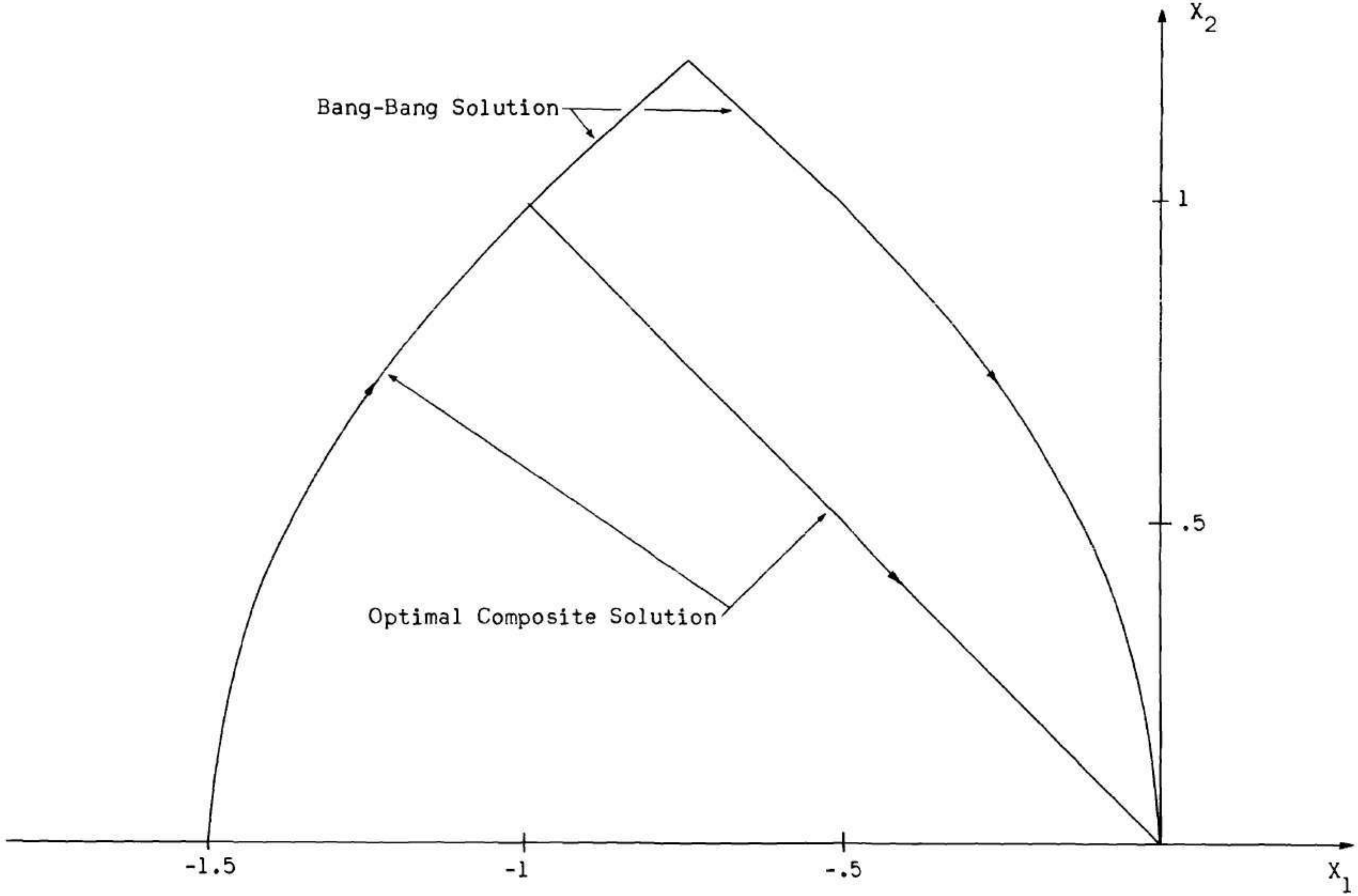

Figure 4. The Optimal and Bang-Bang Solutions.

$$J_{OPT}[u] = 1.5666 \ldots ,$$

and

$$J_{B-B}[u] = 1.667 .$$

This singular problem is normalized in the following manner. Let

$$\bar{J}[u] = J[u] + \int_0^T \alpha u^2 dt = \frac{1}{2}\int_0^T (X_1^2 + X_2^2 + 2\alpha u^2) dt ,$$

then

$$\bar{H} = \frac{1}{2}(X_1^2 + X_2^2 + 2\alpha u^2) + P_1 X_2 + P_2 u$$

from which u^* is found to be

$$u^* = \left\{ \begin{array}{lll} +1 & \text{when} & -\dfrac{P_2}{2\alpha} > 1 \\ -\dfrac{P_2}{2\alpha} & \text{when} & \left| -\dfrac{P_2}{2\alpha} \right| \leq 1 \\ -1 & \text{when} & -\dfrac{P_2}{2\alpha} < -1 \end{array} \right\} .$$

For each value of α chosen, the solution trajectory corresponding to u^* can be determined via the Minimum Principle. In Figure 5, several solution trajectories are presented for various values of α. Inspection of Table 1, which contains the value of $J[u]$ for each suboptimal solution, shows that the suboptimal solution approaches the optimal as α approaches zero. Also note that

$$J_{subopt}[u] < J_{B-B}[u]$$

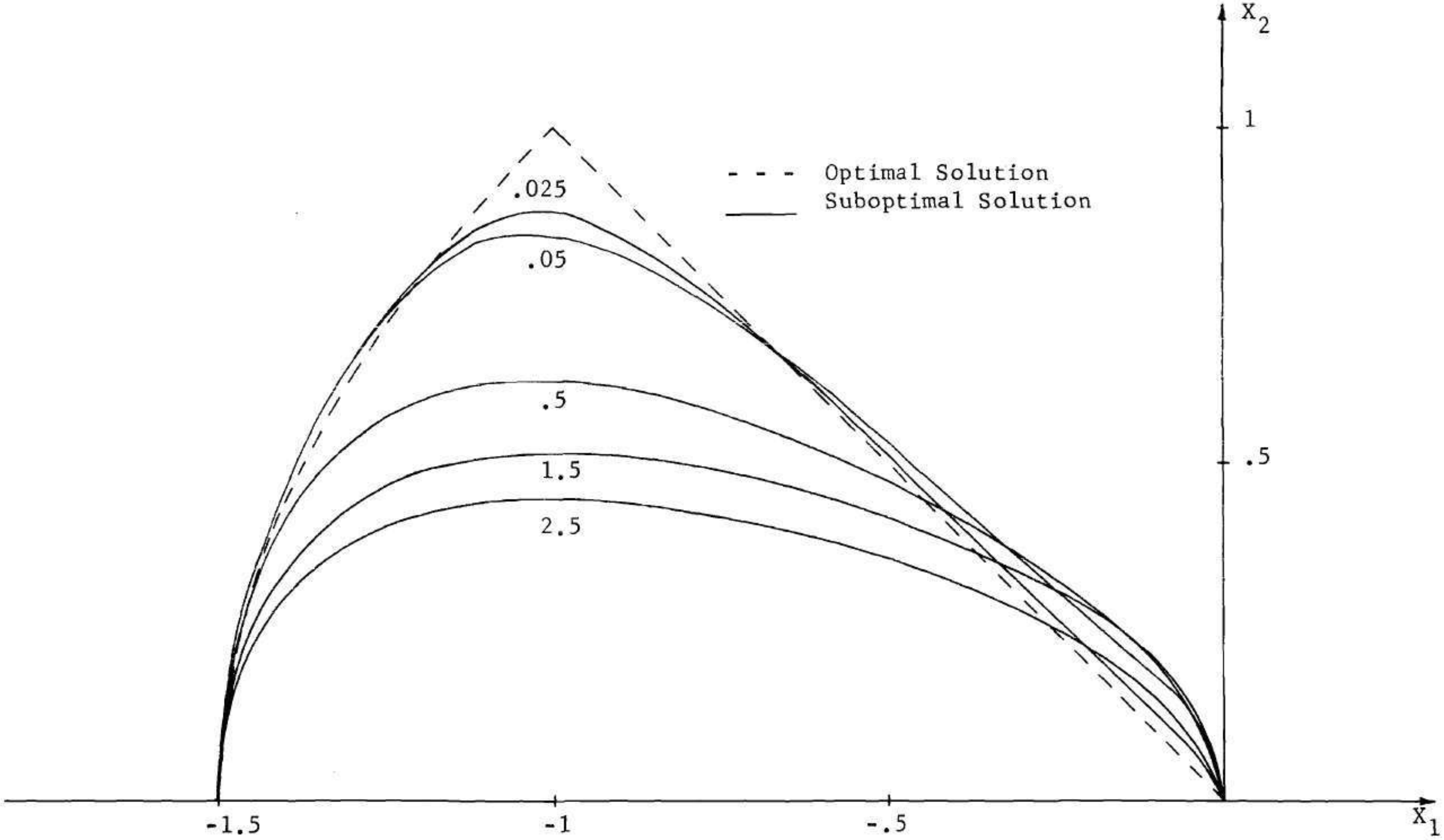

Figure 5. Suboptimal Solutions for Several Values of α.

for

$$\alpha \leq .5 \ .$$

This example shows that the suboptimal solution is superior to the bang-bang solution for small α as well as being an excellent approximation to the optimal solution.

Table 1. The Cost Functional Evaluated Along the Suboptimal Solution as a Function of α.

α	$J[u] = \int_0^T (X_1^2 + X_2^2)\,dt$	T
5.0	2.3425350	6.650
2.5	2.0517385	5.575
1.5	1.8641014	4.745
1.0	1.7641396	4.540
0.5	1.6593561	4.150
0.05	1.5772789	3.800
0.025	1.5698278	4.580

CHAPTER VII

CONCLUSIONS

This dissertation presents a detailed study of several general classes of bounded control problems that are linear in the control variable, i.e., the LOP. The control problems considered consist of linear, nonlinear, or time-varying nth order systems which are expressible in phase variable form, and a cost functional which is to be minimized along the optimal solution trajectory. The quadratic LOP is shown to be singular by deriving all the singular controls and the corresponding fixed- and free-time singular trajectories. Several new results are obtained which enable a better understanding of the singular problem. Among these results are the "feedback cancellation" property of the singular controls and the mechanism of pole-zero cancellation required for the existence of the free-time singular arcs.

It is shown that the singularity of the LOP considered is completely independent of the system dynamics and solely a function of the performance index. Normality is established for several cost functionals while the quadratic cost functional is shown to lead to the singular problem.

Singularity of the LOP with the quadratic cost functional is established by deriving <u>all</u> the admissible singular control functions. The singular control is characterized by its "feedback cancellation" property, i.e., the singular control actually cancels out the original dynamics of the system -- which may be linear, nonlinear, or time-varying-- and substitutes a set of linear stationary dynamics. Therefore, under the application of the singular control, the nonlinear and time-varying systems become linear and stationary.

The analytical expression for the singular trajectories is derived as a function of H_0, the constant value of the Hamiltonian along the singular arc. It is shown that H_0 is only a function of the initial state of the singular arc which implies that the state space is dense in singular trajectories.

The fixed-time singular arcs are characterized by the presence of at least one pair of symmetric eigenvalues in the singular solution which gives rise to the hyperbolic shape of these singular arcs. Necessary and sufficient conditions for the existence of fixed-time singular arcs are given in Theorem III.

Theorem II presents necessary and sufficient conditions for the existence of free-time singular arcs. It is shown that Theorem II can only be satisfied if pole-zero cancellation due to the initial state of the singular arc occurs in order to remove at least one of each pair of symmetric eigenvalues from the singular system. The free-time singular arcs are shown to lie on (n-r)th order hyperplanes in the state space, where r equals the number of poles cancelled.

A technique for normalizing the singular LOP is developed and shown to lead to a suboptimal solution which compares very favorably with the optimal solution.

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